The impact of CAS on our understanding of mathematics education[1]

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Preface

I do not see myself as a member of the CAS research community, but as a member of the mathematics education community. I have had the opportunity to accompany a multi-year TI-92 school-project of my colleague Edith Schneider some years ago. I also had the opportunity and the pleasure of taking part in all the CAME Symposia so far. I learned a lot from the lectures and the discussions at these conferences as well as from different research papers of their participants.

There is no doubt, that these papers are dealing with very important and useful theoretical and practical questions and problems with respect to the use of CAS in mathematics classrooms: Important theoretical considerations and empirical observations about students´ learning behaviour and processes when using CAS, about opportunities and obstacles, about teachers´ behaviour when integrating CAS into the classroom, about changes in the social organisation … and so on.
What I am missing a little bit is the perspective of general mathematics education, the anchoring of the use of CAS in an adequate framework of *general mathematics education for all*.

Such a perspective could give orientations for many considerations and decisions regarding curriculum and CAS-supported teaching and learning. In my following explanations I will illustrate this and focus on operating knowledge

[1] This text is based on ideas and papers of R. Fischer, E. Schneider and W. Peschek, University of Klagenfurt – see references.

and skills, because they seem to be one of the most crucial points in the CAS discussion in the scientific community as well as for teachers.

1 Communication with experts as a principle of orientation for higher general mathematics education

If you ask people “What is mathematics?“ you usually will get the answer “Calculating”. That is, what they have mainly experienced in school mathematics. If someone understands calculating not only as elementary arithmetic operations with numbers, but in a broader sense as algebraic transformations with variables, terms, functions or matrices, as procedures for solving equations, inequalities or systems of equations and inequalities, as determining derivatives and integrals up to developing deductive conclusions, then such a “generalized calculating” according to rigidly formulated rules in fact describes a very important mathematical activity. This “operating by rules” leads to new information about the given fact and with that allows problem solving as well as proof.

But mathematics *education for all* does not mean to learn only mathematics which is important for mathematicians – general mathematics *education* does not mean students are to become a “mini-mathematician”. This widely held misconception leads – at least in Austria – to dilettantism and a threefold crisis of school mathematics (cf. Heymann 1999, p. 147):
The *crisis of acceptance* (students do not perceive mathematics as important for their own life), the *crisis of legitimation* (it’s difficult to legitimate such a mathematics as being necessary for all students) and a *crisis of efficiency* (look for example at the poor results in TIMSS or PISA).

I will start my considerations about *mathematics education for all* with the following socio-philosophical based *thesis*:

The functionality of our highly differentiated, democratic society, built up on the division of labour, is essentially based on emancipated contact with highly specialized expert knowledge. As mature, responsible citizens we are constantly confronted with statements made by experts, which we then must assess and judge in order to be able to make (our own) decisions. As a rule we will rely on the professional correctness of these expert statements, yet do need to judge their importance for ourselves and for the good of the community. As laypersons we must be in the position of being able to ask the experts the right questions, to assess their answers, and to draw our own conclusions. That requires a highly developed communication between experts and laypersons concerning the communicative-social dimension as well as the dimension of contents and concepts. (cf. Heymann 1996, p. 113; Fischer n.d.)

Roland Fischer, a colleague in our department, addresses these considerations by primarily focusing on the aspect of the contents and concepts of communication between experts and laypersons. He proposes that those persons who have attended institutes of higher learning (high schools and vocational high schools), ”the more highly educated” in particular, should be able to understand and to explain the experts’ statements in an understandable fashion and judge their importance. He suggests that such an “ability to communicate with experts and with the general public “is to be taken as a “principle of orientation“ for determining the curriculum at schools of higher education (Fischer n.d., p. 3).

Fischer identifies the following three *fields of competence* as those which are to be acquired for every subject: *basic knowledge* (notions, concepts, forms of representation), *operative knowledge and skills* (solving problems, proofs, in general: generating new knowledge), *reflection* (possibilities, limits and meaning of concepts and methods).

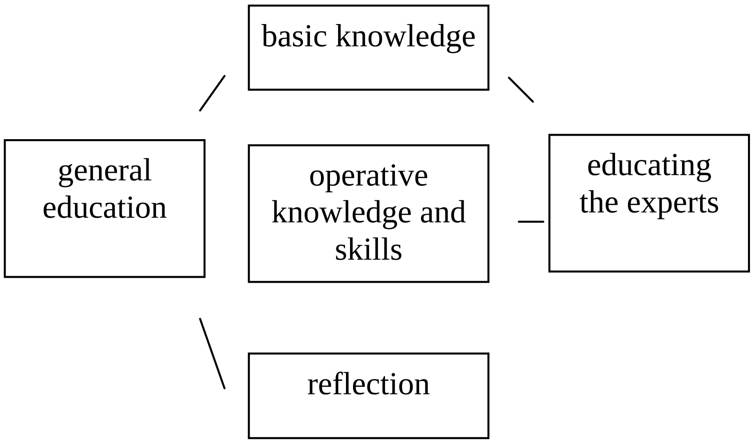


Figure 1: Fields of competence (acc. to Fischer n.d., p. 5)

While the experts in particular have to be competent in the first two fields, Fischer considers the fields of *basic knowledge* and *reflection* to be particularly important for the generally educated layperson (cf. Figure 1). Basic knowledge “is a prerequisite for communicating with experts“, reflection “is necessary for judging their expertises“ (Fischer n.d., p. 5).

As an example, you can think of the communication between a judge and an expert in a trial. The judge – as a generally educated layperson – has to be able to ask the appropriate expert the relevant questions (i.e. about the speed of the car involved in the accident), maybe he has to give additional information to the expert (about the braking distance, the type of the car, the road conditions, etc.), he normally will rely on the expert´s calculation (maybe he will ask the expert, if he took into account, that it was raining and the car was overloaded), but in any case he has to be able to interpret and to judge the expertise and to deduce adequate consequences.

Fischer points out that his classification should not be taken as an absolute. Neither should the experts be relieved of their responsibility of viewing that which they are doing in a self-critical manner, nor can doing operations be completely removed from the framework of a (general) mathematics education.

However, the focus and profiles for experts and laypersons do clearly differ.

The consequence for the teaching of mathematics for all is, in short:

”The reduction of expectations with regard to operations and an increase in the expectations with regard to reflection.“ (Fischer n.d., p. 6)

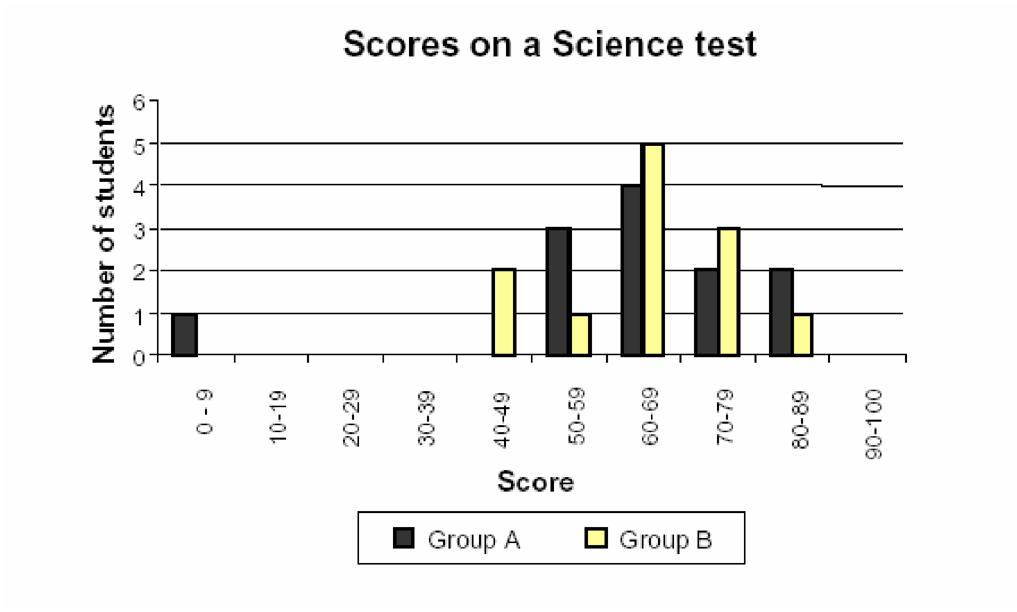
Of course, this position requires strong modifications concerning the contents of mathematics instruction and the tasks treated in mathematics classrooms. I provide an easy example below.

Traditional – reading a diagram, calculating

TEST SCORES

The diagram below shows the results on a Science test for two groups, labelled as Group A and Group B.

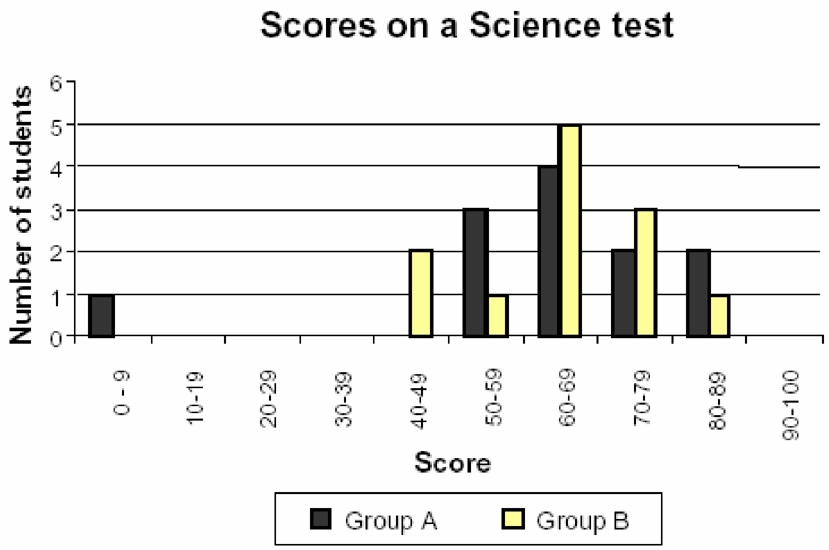
Calculate the mean score for Group A, for Group B and for both groups together!



PISA 2003 – interpreting, reflecting, arguing

TEST SCORES

The diagram below shows the results on a Science test for two groups, labelled as Group A and Group B.
The mean score for Group A is 62.0 and the mean for Group B is 64.5. Students pass this test when their score is 50 or above.



Looking at the diagram, the teacher claims that Group B did better than Group A in this test.
The students in Group A don't agree with their teacher. They try to convince the teacher that Group B may not necessarily have done better.

Give one mathematical argument, using the graph, that the students in Group A could use.

I am far from pretending that PISA tests are founded on our concept of general mathematics education. PISA-mathematics is based on a specific interpretation of "Mathematical Literacy" and the test-items touch only a few aspects of this mathematical literacy. But with regard to our concept of general mathematics education, it is interesting to observe that in all the 84 test-items of PISA-mathematics 2003, you can hardly find items which demand algebraic transformations, and only very few items where elementary arithmetic transformations are needed. (And in addition: PISA allows use of any calculator you like up to CAS!)

This observation is very interesting for me in a twofold respect:

First, it is interesting that PISA is going almost without testing operative abilities – even for 15-year old students (our concept of general mathematics education is proposed more for high-school students). Second, I am astonished that the governments of all OECD-countries accept this test as a highly relevant, essential and expressive assessment of students´ mathematical competencies, since this test is far from the operative knowledge and skills mainly taught in traditional classrooms.

This can be seen as an inconsequence of our educational system and politics, but in every case it is to be seen as a strong signal of the political and social expectations or demands, and in Germany and Austria it has clear consequences for the National Standards for School Mathematics. For example:

Austrian Standards for School Mathematics – modelling, reasoning

AVERAGE HEIGHT

For a school-statistics you are asked to find out the average height of the students of your class.

What average value (mean, median, ...) would you pick out? Give the reasons for your decision!

2 Using CAS is communication with experts

Mathematics teaching, which follows our concept of general mathematics education (or the concept of mathematical literacy used in PISA test-items), will focus on basic knowledge such as notions, concepts, different forms of representations, areas of applications, as well as on interpretations and reflections. Operational work occurs rather seldom and is restricted to elementary transformations. Despite the fact that CAS can help a lot in developing basic knowledge and reflected (conceptual) understanding – as is shown by many colleagues from the CAS research community – such mathematics teaching seems not to be an inviting field for using CAS because of the lack of operative work.

With regard to the work of my colleague Edith Schneider (2002, also Schneider & Peschek 2002) I take a different view. According to our concept of general mathematics education, I assign the field of competence for operative knowledge and skills primarily to the experts. This turns out to be exactly the same field of competence which could be delegated most completely to CAS. In this (narrow) sense we can perceive CAS as a simple electronic model of a mathematical expert.

I admit CAS does not cut a very good figure as a mathematical expert, because it is too limited and rigid in its communication with us users; its basic knowledge of mathematics, its abilities of representation and of interpretation are too insufficient. CAS can sometimes even disappoint us in the operative field. CAS can never become a substitute for human experts (and particularly not for teachers). But in using CAS, in the communication between human being and machine, elements can be seen that are quite significant for communication between laypersons and human experts.

A successful and profitable interaction with human as well as electronic mathematical experts requires

- the willingness and ability to ask the “right“ questions, to be precise when formulating one’s own questions and considerations and to present them in a form which can be interpreted by the expert;

- quite exact conceptions of the possibilities and limits of the mathematical expert's knowledge and skills, and of the range of validity of mathematical expressions;
- wide basic knowledge of mathematics (especially knowledge about important mathematical forms of representation);
- a verification as well as an appropriate interpretation and assessment of the answers given by the expert.

Whenever CAS users (students) are working in corresponding forms of interaction, something else is happening. There might arise the transmission of the answers provided by CAS to other laypersons (students), the discussion of these answers among the laypersons and the negotiation processes of their interpretation, and justification of any further questions to the expert. All in all, these are essential components of what Fischer calls "communication with the general public".

For the reasons briefly outlined here I see the use of CAS as a simple but very useful model for the communication between mathematical experts and laypersons. Reflection on it can and should be used didactically and pedagogically in mathematics classrooms.

Two additional remarks:

1. In our concept of general mathematics education you can omit problem solving, which demands higher developed operative abilities. (We are sceptical that problem solving is a realistic and sensible aim of general mathematics education at the high school level.) Of course, such problem solving becomes easier (and more realistic) with the help of the expert CAS.

2. In the traditional mathematics classroom teachers are usually the only experts (maybe beside textbooks). So the communication between teachers and students is an important model for the communication between experts and laypersons.
I don't want to question this important role of the teacher. But I would like to point out that, at least in Austria, the responsibility for the operative work is seen to be up to the students rather than up to the teachers (look at the tasks given to students!), while teachers feel responsible to help the students with modelling, with selecting suitable representations, with interpreting, reflecting and judging the results. This casting is in contradiction to our concept (of the role of experts and laypersons), while the communication between students and CAS fits exactly into our concept.

3 The damned "black boxes"

In our concept of higher mathematics education operative abilities and skills are not the crux and can largely be delegated (outsourced) to mathematical experts (as CAS) and/or be included in the communication by these experts. In many cases it is neither necessary nor sensible (and sometimes not even feasible for the students) to know the internal working of such outsourced operative modules (which are then frequently called "black boxes").

I know, that there are many colleagues, in particular mathematicians, who strictly contradict this view.

"The main point that we have tried to establish, through several examples, is that we cannot reasonably use a CAS as a black box, in particular in the classroom." (Elbaz-Vincent 2004, p. 63)

"It would be disastrous for the future of mathematics if the insights and techniques that can be taught and learned … would be ignored because the area is "trivialized" and, therefore, … all problems in this area can be handled by the available systems." (Buchberger, 1989, p. 4)

Buchberger´s arguments, as well as the examples given by Elbaz-Vincent, refer strongly to a mathematical viewpoint (demanding that mathematics at school has to produce young “mini-experts” to assure the future of mathematics), but not to an *educational* one.

While understanding these arguments as a mathematical viewpoint, I am a little bit astonished by them. An understanding application of black boxes is of great importance not only in our daily lives, but also in the sciences. This is valid in mathematics particularly for scientific-theoretical and socio-philosophical reasons.

In mathematics we constantly work with the method of *outsourcing*. This occurs not only in elementary procedures such as division algorithms, but also in the more complex notions and procedures up to and including proofs. One needs not know why the division algorithm being applied works in order to get the right answer when dividing; one needs not bother with the logical reasoning behind an equivalence transformation when using such a transformation to solve an equation; and one needs not recognize the basics of Set Theory or the concept of function in order to succeed in Calculus when calculating a derivative. One uses many of these mathematical concepts and procedures as *compromised bits of knowledge* (modules) within mathematics, of which one needs to know very well the prerequisites, effects, ranges, limits and the “interfaces” to the outside, but not their internal workings, in order to be able to apply them correctly and in a sensible way.

In a certain way outsourcing occurs in mathematics whenever one abstracts relationships from the (reference) context and presents them with symbols, thus outsourcing the problem in the formal-operative system of mathematics. This outsourcing allows operations to be carried out on a syntactic level, without having any correspondence to the reference context and not being bound to it (therefore, in a certain sense, “without understanding“). The results calculated within the formal system can be interpreted in the original context and result in the solution of the investigated problem. Such an approach reduces the complexity of the problem, it allows for economic thinking as well as for solutions and methods of solving which otherwise, without the possibility of outsourcing in the formal system, would either not be found or not be so simple to find.

Such an *outsourcing* is something *genuine for mathematics*; it is one of the characteristics of mathematics and it is an essential basis for its performance ability and efficiency. Computers and CAS are for the moment just the last step in the development; they are an extension (in regard to transformations by rules) and perfection of outsourcing made possible by materializing abstract mathematical situations in machines.

One can immediately establish analogies between these scientific-theoretical considerations and considerations on the *socio-philosophical level*[2]. In our high-tech and high-economic society with its organisation by division of labour, the constant use of black boxes has long become an indisputable necessity. There mathematics is taking on a special role with increasing social importance.
Mathematics is relatively secure, socially accepted, codified knowledge, which notably allows for a separation between understanding and doing ... (it) owes its high social relevance to the fact that, in utilizing outsourcing, it even works when the user has no idea anymore as to why.

I see it as a “paradox of mathematics education” that we make a genuine effort

[2] A more detailed and encompassing discussion on this matter can be found, for example, in Fischer 1991; Maaß & Schlöglmann 1988; Peschek 1999.

to make our students understand mathematics, although its social value is based on the fact that mathematics works without understanding. (Ask your students!)

Thus, an elaborate image of mathematics, of its characteristics, its ways of thinking and working, and its socio-cultural relevance, would include outsourcing as an important scientific-theoretical and socio-philosophical aspect. This should be experienced as a fundamental characteristic of mathematics and be understood as a constitutive aspect of its social relevance. CAS, and computers in general, can serve as very notable, illustrative example and models for outsourcing mathematics, and like these simplify reflection and discussion of this basic characteristic of our science. I am convinced that such reflections are much more important for the general mathematics education of our students than bothering them with "trivialized" knowledge and skills.

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