

Notes about Reflection, Operative Knowledge, and Execution

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If we want to capture the changes that Computer Algebra Systems (CAS) have brought into mathematics education, we need to have a global perspective regarding the process of teaching and learning mathematics. Mathematics educators have been challenged to identify and explore such global perspectives. We bring here several approaches on which we base our analysis of teachers solving problems in a CAS environment. Heid (2002) emphasized the role of automated production of representations and translations between them as a key to our thinking about the impact of CAS on mathematics learning. Peschek and Schneider (2001) introduced the didactical principle of *outsourcing* operative knowledge and operative skills as a didactical principle of CAS use. They regard operative knowledge as means to generate new knowledge (Peschek, 2005).

An anthropological approach, based on the work of Chevallard, has been incorporated by Lagrange (1999) and Artigue (2002) in their research of computer algebra in mathematics education. According to Chevallard (1992) the didactics of mathematics is the part that is concerned with mathematical knowledge and its functioning in learning mathematical topics. A topic can be characterized by its practices, or rather by its theory of practices, i. e. a *praxeology*. A praxeology of a given topic can be described by a consistent set of four components: *tasks* (in which the topic is embedded), *techniques* (for accomplishing the tasks), *technology* (a discourse justifying the techniques), and *theory* (a proposed structural basis for the technological discourse). Pierce and Stacey (2004) conclude their empirical study saying that "Using a CAS to do mathematics requires traditional mathematical knowledge; knowledge of the machine and ... it also requires a constant interplay of mathematical knowledge and knowledge of the technology". (Pay attention to the different meanings of the word "technology": in Chevallard's point of view, technology is the discourse (logos) about the technique, Pierce and Stacey's technology is Lagrange and Artigue's artifacts.)

In a previous paper, entitled *Instrumented Techniques and Reflective Thinking in Analytic Geometry* (Zehavi and Mann, 2005), we compared a traditional solution

with a CAS-based solution of the problem of finding the locus of points of intersection of perpendicular tangents to a given ellipse (this locus is called the *director circle* of the ellipse). The steps of the solution were analyzed in two levels: the *Execution* level and the *Reflective Thinking* level. The basic components considered in the *Execution* level in problem solving are: constructing a mathematical *model* for the problem, *manipulations* within the model to obtain results, *interpretation* of the results in the context of the problem, and *representations* (graphical or symbolic) of the model, or the manipulations, or the interpretations. The *Reflective Thinking* is a meta-cognitive level in which we identify four categories: selection of *techniques*, *monitoring* the solution process, *insight* or ingenuity, and *conceptualization*. Our analysis indicates that in traditional solutions insight and conceptualization are prior to the execution steps, while in CAS solutions the reflection steps are inseparable from the execution steps. A consequent implication is that advanced problems that have been traditionally reserved for those few students gifted with mathematical intuition, can now be accessed effectively by a greater population with appropriate instruction by the teachers. The data was obtained in workshops for in-service teachers as part of the formative development process of a resource e-book for teaching Analytic Geometry with CAS (we use *Derive*). Observation of teachers' exploration of the director circle of the hyperbola enabled us to deepen our analysis by focusing on (a) the role of Operative Knowledge (OK) as a mediator between Reflection and Execution, and (b) the evolution of the OK in the CAS era. We attempt to elaborate on the technological discourse in terms of the OK.

CAS-based Operative Knowledge

A group of 18 teachers participated in a workshop in which they solved problems in analytic geometry using *Derive*. All of them were familiar with the software, and most of them used it in their work in various ways; twelve of them reported that they had fair experience in using CAS while the other six teachers reported that they had extensive experience. The teachers were asked to find the locus of intersection points of perpendicular tangents to the hyperbola defined by the equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$. First we describe here a typical solution process of the teachers who had fair experience. As observed in the previous case study the teachers began by *modeling the problem* (Execution step, based on traditional OK):

The equation of a line (not parallel to the vertical axis) that passes through the point whose coordinates are (p, q) is $y = mx - mp + q$. By substitution we get an equation for the x values of the intersection points of the hyperbola and the line:

$$\frac{x^2}{9} - \frac{(m \cdot x - m \cdot p + q)^2}{4} - 1 = 0$$

Next they *selected a familiar technique* (Reflection Step, traditional OK) for simplifying the equation to obtain the well known *model* of a quadratic equation (with the requirement that $\Delta = 0$).

Looking for values of m that satisfy the equation:

$$(18 \cdot m \cdot (q - m \cdot p))^2 - 4 \cdot (9 \cdot m^2 - 4) \cdot (9 \cdot (m \cdot p - 2 \cdot m \cdot p \cdot q + q^2 + 4)) = 0$$

Outsourcing *Derive's* manipulations (Execution step using new OK) yield two algebraic solutions for m :

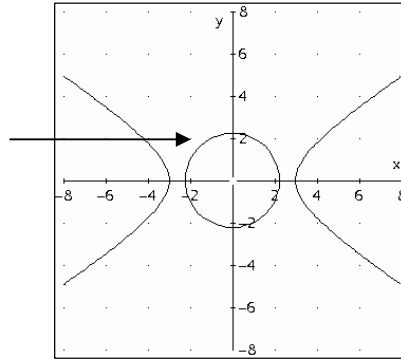
$$m = \frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2) - p \cdot q}}{9 - p^2} \vee m = \frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2) + p \cdot q}}{p^2 - 9}$$

Conceptualization, namely translating the necessary and sufficient condition (if and only if) for lines to be perpendicular, resulted in an equation (Reflection step).

The product of the slopes of two orthogonal lines is -1 , thus:

$$\frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2) - p \cdot q}}{9 - p^2} \cdot \frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2) + p \cdot q}}{p^2 - 9} = -1$$

Reflection step: the 'strange' equation called for *monitoring*. It has been done by outsourcing the graphic facility as shown in the figure (new OK, which is different from the 'calculation' type of operative knowledge).



Is this a circle? Let's 'simplify' the equation.

Execution step: Manipulating the above equation i.e. outsourcing symbolic calculation (new OK) resulted in a 'simple' equation.

Let's plot (the equation):

$$\frac{z}{p - 9} + 4 = -1$$

Interpretation and another symbolic *representation* (Execution/Reflection step) concluded the solution.

Simplification of this equation yields $p^2 + q^2 = 13$. This is an equation of a circle, usually written $x^2 + y^2 = 13$.

The sequence of steps indicates that in the presence of the technology reflective thinking, operative knowledge, and execution became intertwined. We see in the above that the links that the OK created between the execution and reflection were mainly observed in the manipulation and in selecting techniques. At the beginning of the solution process the teachers used the software as an assistant to implement traditional operative skills; gradually their actions relied on new OK that becomes available by the CAS.

Now we will describe how the teachers with a good mastery of *Derive* worked on the same problem. They began their solution by modeling the problem in a different form (Execution step based on new OK):

We write a system of two equations (in the notation of the software). Outsourcing the solutions for (x, y) of the system of equations $\frac{x^2}{9} - \frac{y^2}{4} = 1 \wedge y = mx - mp + q$ resulted in two pairs of solutions:

$$\left(\begin{array}{l} x = \frac{3 \cdot (2 \cdot \sqrt{(m^2 \cdot (p^2 - 9) - 2 \cdot m \cdot p \cdot q + q^2 + 4)} - 3 \cdot m \cdot (m \cdot p - q))}{4 - 9 \cdot m^2} \wedge y = \\ \frac{2 \cdot (3 \cdot m \cdot \sqrt{(m^2 \cdot (p^2 - 9) - 2 \cdot m \cdot p \cdot q + q^2 + 4)} - 2 \cdot (m \cdot p - q))}{4 - 9 \cdot m^2} \end{array} \right) \vee \left(\begin{array}{l} x = \\ \frac{3 \cdot (2 \cdot \sqrt{(m^2 \cdot (p^2 - 9) - 2 \cdot m \cdot p \cdot q + q^2 + 4)} + 3 \cdot m \cdot (m \cdot p - q))}{9 \cdot m^2 - 4} \wedge y = \\ \frac{2 \cdot (3 \cdot m \cdot \sqrt{(m^2 \cdot (p^2 - 9) - 2 \cdot m \cdot p \cdot q + q^2 + 4)} + 2 \cdot (m \cdot p - q))}{9 \cdot m^2 - 4} \end{array} \right)$$

A Reflection step involves the selection of an unfamiliar technique and conceptualization which relies on new OK:

If we are looking for tangents to the hyperbola we need have only one solution, meaning finding for what values of m the expression under the square root is equal to zero.

$$\text{SOLVE}(m^2 \cdot (p^2 - 9) - 2 \cdot m \cdot p \cdot q + q^2 + 4 = 0, m)$$

$$m = \frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2)} - p \cdot q}{9 - p^2} \vee m = \frac{\sqrt{(9 \cdot (q^2 + 4) - 4 \cdot p^2)} + p \cdot q}{p^2 - 9}$$

(Note that these are in fact the solutions of $\Delta = 0$ in the method used by the other teachers.)

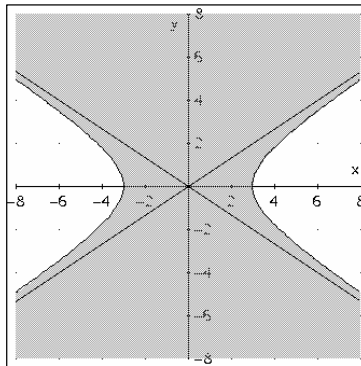
Reflective thinking led to further conceptualization (supported by new OK).

The existence condition:

$$4 \cdot (9 \cdot (q^2 + 4) - 4 \cdot p^2) > 0$$

Meaning that, a tangent to the hyperbola passes through a point if, and only if, the point is either on the hyperbola or outside the hyperbola (i.e. in the colored region of the figure).

This evolving new OK made the teachers note that the asymptotes are included in the outer region of the hyperbola.



Conceptualization (Reflection step): translating the necessary and sufficient condition (if and only if) for lines to be perpendicular, resulted in an equation, as the other teachers did. However, Manipulating the above equation i.e. outsourcing symbolic calculation (new OK) for Execution resulted in a 'simple' equation, followed by an unfamiliar instrumented technique (considered as a 'trick' for simplifying an equation with two variables that contains fractions:

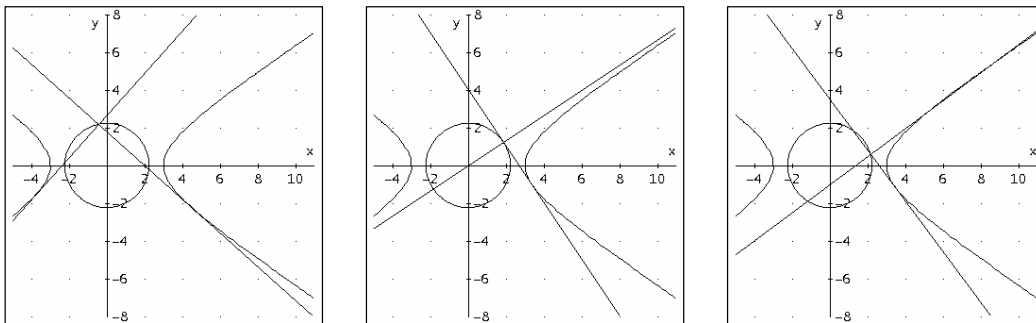
$$\frac{z^2}{q^2 + 4} = -1$$

$$\frac{z^2}{p^2 - 9} = -1$$

$$\text{SOLVE} \left[\frac{z^2}{q^2 + 4} = -1, [p, q] \right] = (p^2 + q^2 = 5 \wedge p \neq -3 \wedge p \neq 3)$$

The pragmatic value is clear. The epistemic value was expressed by the questions teachers asked regarding the existence conditions ($p \neq \pm 3$) given by the computer. Interpretation of the results led to plotting the director circle as well as animating pairs of tangents as an implementation of the result. The instrumented technique needed for this task involves the use of a slider bar to view, in a dynamic way, pairs of tangents that intersect in a point $T = (p, 5\sqrt{5-p^2})$ on the director circle whose equation is

$x^2 + y^2 = 5$. Again, the evolving new OK motivated teachers to explore perpendicular tangents at various points on the director circle, and in particular the intersection points of the asymptotes and the director circle (the middle figure), realizing that these are special points! The asymptotes are the locus of points, which are not on the hyperbola, and through which only one tangent to the hyperbola passes. We should mention that the origin of the axes provides a new situation: no tangent to the hyperbola passes through the origin.



Technological discourse

It is clear that mathematical Operative Knowledge evolved in the last decades. In the latter solution as well as in other studies (Zehavi, 2004, Pierce and Stacey, 2004) we observed awareness of the ways that the software utilizes symbols in algebraic manipulations. Teachers who have developed a good mastery of the software took advantage of the technological awareness and developed new symbolic instrumented techniques. The easiness of making CAS execute various algorithmic tasks is the engine that drives the evolution of Operative Knowledge.

Moreover, in the past, we could not plot implicitly (manually or otherwise) therefore we had to be told by an outside authority what is the geometric shape (sometimes we knew it simply because the problem appeared in a certain chapter in the textbook), and workout the algebraic manipulations manually. According to Heid (2002), the study of translation between representations produced by CAS is part of the theoretical framework of our research..

We maintain that involving CAS in the Execution of the tasks monitored by Reflective Thinking acts on our Operative Knowledge that must adapt itself to its evolving role. In return, the evolution of Operative Knowledge has obvious influences on the ways of Reflective Thinking. In the past no one thought of implicit plotting

because it simply could not be done manually. Our experience with teachers shows that after a little experience people use it extensively (new OK) because it is so powerful on the one hand, and so easy to get from CAS on the other hand. Furthermore, we get used after a while to plot equations that we did not bother to deal with in the past because they were inhibiting.

Chevallard claims that mathematical objects arise from the practice of a given institution. Thus, his anthropological approach could be used to make comparisons between the traditional and emerging practices of mathematics teaching and learning. In shaping the resource e-book in Analytic Geometry we included new *tasks* and new *techniques*. Some of the techniques have become routine for teachers in our institution but they are afraid that these techniques can become "de-mathematized" by students. Therefore, we try to make explicit the epistemic value of the techniques. Artigue (2002) stresses the *epistemic* value of techniques (in addition to their *pragmatic* value), as they are a source of questions about mathematical knowledge. The French researchers combine Chevallard's *technological* and *theoretical* components. We feel that our interpretation of the Operative Knowledge gives some new meaning to the *technological discourse*. Operative Knowledge enables us to realize the potential of the technology. This prompts new developments of reflective thinking. Thus, the evolution of OK is reflected in evolution of our reflective thinking. At the same time, OK is a necessary condition for outsourcing the technology's mathematical work in an efficient and productive way. Actually, the productivity increased in a remarkable proportion. We believe that this will affect the curriculum; for example, see (Dana-Picard, 2004). In the situation presented in this paper the exploration of the "outside" of a hyperbola can be added to the traditional curriculum and the "director circle" can be used as an additional unifying point of view for conic sections. The ideas expressed above propose some initial elements for what we believe to be a theory of practice of teaching mathematics with CAS.

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