DESIGNING TASKS FOR CAS CLASSROOMS: CHALLENGES AND OPPORTUNITIES FOR TEACHERS AND RESEARCHERS

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Using a semiotic framework I isolate three key components of a CAS–based mathematical task: construction of signs, experimentation and transformation of signs, and interpretation of the transformed signs. Within each of these components, the design of a CAS–based task has implications for the production or interpretation of mathematical signs. The construction of appropriate CAS–based signs is often problematic for students and the level of guidance in this regard is a challenge for the teacher. At the same time, complexities around the construction of signs may open up new possibilities for task design. Furthermore, certain information in a task statement may profoundly affect students' experimentation and interpretation; the extent to which such information is provided or withheld provides opportunity and challenge for the teacher.

INTRODUCTION

I use a semiotic framework to deconstruct a CAS–based task into its components. I then illustrate how this framework illuminates aspects of the task design, which may promote or hinder mathematical activity. Throughout the talk, I assume that a CAS–based task involves the use of both CAS and paper-and-pencil (hand) work.

THEORETICAL FRAMEWORK

I regard mathematics as a semiotic system, that is, as a system of signs. I assume-Ernest's [1] formulation of- mathematics as consisting of three components: a set of signs which may be written or uttered or encoded electronically, a set of rules for sign production and a "set of relationships between signs and their meanings embodied in an underlying meaning structure" [1, p. 70]. Within a semiotic framework, the idea of the mathematical object and the written, spoken or encoded inscription are mutually constitutive. Neither can exist without the other and both evolve with each other. This view of mathematics is particularly relevant for task design since it implies that, for the learner, mathematical concepts and the production of signs are mutually constitutive.

Furthermore I use Peirce's notion of mathematical reasoning [2] to deconstruct a CAS–based task into its major components. C. S. Peirce (1839–1914), an American mathematician and founding father of semiotics, proposed that all thinking is performed upon signs of some kind or other, imagined or perceived. Peirce regarded signs not only as a means of signifying or referring to an object, but also a "means of thought, of understanding, of reasoning and of learning" [2, p. 45] Peirce argued that all deduction and mathematical reasoning involves the **construction** of an appropriate sign or diagram, **experimentation** on this set of signs through manipulations and transformations of signs (written, spoken or imagined), and observation (which of necessity includes **interpretation**) of the transformed set of signs.

COMPONENTS OF TASKS

Peirce's categorization of mathematical reasoning is particularly useful for isolating key components of mathematical tasks and examining their implications for teaching and learning. In the context of a CAS–based task, these activities assume a very particular form and function.

I illustrate the illuminating power of Peirce's categories by using these categories to critique a CAS–based task, which I adapted from an undergraduate mathematics textbook [3] in 2007. This task was part of an assignment given to a class of 202 first-year university mathematics students of varying abilities and computer experience. The assignment was intended to introduce students to the concept of the Maclaurin polynomial before the students had been introduced to the concept in regular mathematics lectures. The purpose of the particular task (see Figure 1) was to help students understand the notion of an interval on which the Maclaurin polynomial approximated the original function by a certain amount (e.g., 0.1 units) and how to find this interval.

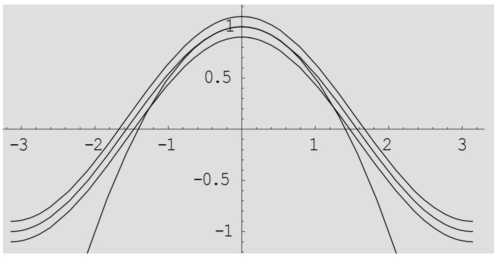

Figure 1. Maclaurin polynomial task

In a previous task, students were required to generate the second order Maclaurin polynomial $p(x) = 1 - \frac{1}{2}x^2$ and to plot a graph of $p(x)$ and $f(x) = \cos x$ on the same set of axes (see [4] for further details).

> Determine the values of x for which the quadratic approximation $p(x)$ found previously is accurate to $f(x)$ within 0.1. [Hint: Graph the functions, $f(x) = \cos x$, $p(x)$, $y1 = \cos x + 0.1$ and $y2 = \cos x - 0.1$ on a common screen.]

Construction of sign

At some point, most CAS–based tasks require students to construct a suitable mathematical signifier (a set of inscriptions, symbols). This may be a representation of a mathematical object, such as a graph or the definition of a function, or it may be a statement concerning a procedure, such solving an equation.

With regard to the exemplar task, my expectation was that students would represent the situation graphically as per the hint and use this result to gain epistemological access to the problem. However only 58% of the class (117 of 202 students) successfully constructed an 'illuminating' plot of the four graphs (often after much trial and error) by using an adequate domain for the graphs; 26% of the class generated the graphs using a less than optimum domain (although 9% of the class were still able to solve the problem despite domain difficulty); 17% were unable to produce a reasonable plot. Thus approximately 34% of this class were unable to construct appropriate signifiers for this task.

Although we may think that first–year mathematics students should be able to easily find an optimum domain (or window), this did not happen for approximately one-third of my class. Possibly this was due to a lack of familiarity with *Mathematica* (these students had a one–hour *Mathematica* tutorial every two weeks and 13% of the class were not computer literate on entry to university). In this pedagogical context, I suggest that the task would have better achieved its intended aim (regarding the notion of an approximating polynomial) had I explicitly suggested a reasonable window. Certainly in pedagogical contexts where students are better rehearsed in using CAS to sketch graphs, such an explicit instruction may not be necessary. This example illustrates see how a small change in the task design may have powerful implications for the quality of the learning that might result.

A brief mention of syntactical intricacies which students encounter when using CAS is necessary. For example, different usages of the equal sign imply different types of equivalence in *Mathematica.* To define a function, $f(x)$, as say, $\cos x$, one enters $f[x\_]$: = Cos[x]; to solve an equation, one enters, say: [Solve[$f[x]$ == $g[x]$, x]; to define a constant, one enters, say, Area = $\pi * r^2$ where r has been defined as radius. Being able to distinguish between these different uses of the equal sign requires a type of mathematical awareness different from the mathematical awareness required in paper-and-pencil work. Inter alia, it requires an understanding of different types of equivalence. Such rarefied knowledge is a crucial part of transforming the CAS into a tool for learning. (Transforming CAS into a tool for learning is one aspect of the process of instrumental genesis, a theoretical framework used by many educators in the CAS world. See [5] for elaboration of this framework.)

Clearly adequate design of CAS tasks requires awareness of the type of hybrid knowledge (mathematical or syntactical) required of a CAS user to construct appropriate mathematical signs. Where necessary the task design should include guidance on such construction. Or the complexity of the different uses of signs can be exploited in the task design. For example, a task may require a student to explicitly distinguish between different uses of the equal sign and to articulate the mathematical significance of that distinction.

Experimentation

Most non-trivial mathematical tasks involve some form of experimentation. I mention two forms of experimentation relevant to CAS here.

CAS tasks resulting in conjectures: One of the most important uses of CAS as a tool of learning is students' use of this resource to generate examples of a particular mathematical construct. From these many examples a student may generate hypotheses of generalised mathematical statements that then need to be proved, probably using paper and pencil. This sort of task is important in several respects. It allows students to move from the particular (various examples) to the general. Furthermore students need to be aware that finding patterns, a common feature of reform type mathematics, is not sufficient to establish mathematical truth. For mathematical certainty, deductive proof (not an inductive conjecture) is required.

Pragmatic versus epistemic values of experimentation: One of the much espoused virtues of CAS is that it can be used as a tool for outsourcing processing power; that is, one can use CAS to perform time consuming and tedious tasks. However this use of CAS as a tool for computation is not without consequences. The French CAS school (see [5]) differentiates between activities that have a pragmatic value and activities that have an epistemic value. Pragmatic values concern the efficiency or productive potential of certain mathematical activities; epistemic values concern the extent to which mathematical activities contribute to an understanding of mathematical objects.

For example, one can use *Mathematica* to find the zeros of polynomial equation $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36$. One enters [Solve $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36, x$]. The computer outputs the zeros in question in the form: $\{\{x\to-4\},\{x\to-3\},\{x\to-1\},\{x\to1\},\{x\to3\}\}$. Clearly this is a very efficient way of solving the equation–it has high pragmatic value, and in, say, a mathematical modelling task it may well be a productive use of CAS (enabling the student to devote her intellectual energies to the modelling task). However, the epistemic value of hand solving an equation will largely be lost in the CAS environment. To solve $x^5 - 10x^3 + 9x = -4x^4 + 40x^2 - 36$ by hand, one might write: $f(x) = x^5 - 10x^3 + 9x + 4x^4 - 40x^2 + 36$. One can then use the Factor Theorem[1] together with polynomial division (and a lot of time) to find that $f(1) = 0, f(-1) =0, f(3) = 0, f(-3) = 0$ and $f(4) = 0$. Hence $x = -4$ or $x = -3$ or $x = -1$ or $x = 1$ or $x = 3$. The epistemic value of such hand work is high; through this experimentation the student may gain insight into the nature of roots of a polynomial, the Factor Theorem, and so on.

Similar to the French school argument about the technical–conceptual cut [5] and as illustrated by the above example, procedural knowledge is intricately linked to conceptual knowledge [6]. As much as one may regard the execution of certain procedures (such as solving a fifth-order polynomial) tedious and mechanical, one may gain conceptual knowledge about the properties and features of the mathematical construct. Thus, although the use of CAS may free the student from cumbersome activities, its thoughtless use may contribute to the underdevelopment of certain concepts, which are often learnt in tandem with procedural activities.

In my exemplar task, the experimentation with CAS was intended to have both a pragmatic aspect and an epistemic aspect. I expected students to use CAS to

[1] Factor theorem: The number c is a root of the polynomial function $y = p(x)$ if and only if $x - c$ is a factor of $p(x)$.

generate a visual picture of the mathematical situation more easily and accurately than with paper and pencil (the pragmatic aspect). One reason why this was not always realised might be partly traced back to many students' problems with constructing an appropriate window of graphs, as discussed above. The epistemic aspect involved the students seeing a visual representation of the second-degree Maclaurin polynomial together with $\cos x$. This was expected to give students insight into a Maclaurin polynomial—an insight not easily achieved through by-hand work.

With respect to task design, I could have attempted to increase the epistemic value of the task by withholding the hint (thereby requiring students to actively reflect on the mathematical situation). But I suspect that, without the hint, many students would have been unable to move forward and much epistemic value would have been lost for these students. This illustrates how the provision or withholding of certain information in a task statement may enable or inhibit experimentation and profoundly affect the epistemic value of the task.

Interpretation

In doing any mathematical task, a learner needs to interpret various signs. The interpretation of CAS–based signs is often different from the interpretation of traditional mathematics signs. This interpretation may involve a CAS user simply being aware of certain conventions in the output format of the CAS. In the example mentioned above, the user needs to know that output '$x \to 4$' is equivalent to the paper-and-pencil line '$x = 4$'. Or it may involve a more subtle mathematical awareness. For example, CAS may produce the same simplified expression for two non-equivalent algebraic expressions. Asking, say, *Mathematica* to Simplify $(x^2 - 1 / (x - 1)$ yields the output $x + 1$ with no mention of the restriction, $x \neq 1$, and so the user needs to interpret the output with a mathematically critical eye.

Interpretation of graphical representations in a computer environment also entails its own challenges and opportunities. Students' work with my Maclaurin polynomial illustrates this dichotomy. In their work with the Maclaurin polynomial task, two students use an (illuminating) plot of the four graphs that they had generated to help them interpret the task. Using visual inspection they make the crucial observation that the polynomial, $p(x)$, intersects only $\cos x - 0.1$. In this example, the CAS–generated graphs provide the students with epistemological access to the mathematics of the task. Indeed, one student exclaims: "**I think I can see. I can see what is going on**. 'Cause this, no. It's the graph of $\cos x - 0.1$ (*pointing to screen*). It goes here (*pointing to graphs of* $\cos x - 0.1$.) It intersects with the graph of $p(x)$ and it goes here". The pair of students then proceeds to correctly solve the task. In solving tasks, some CAS features generate challenges for learning, whereas some CAS features provide opportunities for learning.

But 'seeing' does not necessarily lead to an appropriate interpretation of the mathematical content. Specifically, although many students were able to identify the points of intersection in the Maclaurin polynomial example, only approximately one-third of them interpreted the points of intersection of the two graphs as the endpoints of an interval (which is what they were required to do). For these students it was as if use of CAS becomes the aim of the task rather than a means to solving the problem given in the task. This is our challenge.

Thus we see, as with traditional mathematics, the interpretation of CAS−based tasks may be unexpected. The extent to which a teacher should guide students in their interpretations depends very much on the particular task and the teacher's intentions. This is an area worthy of further research.

CONCLUSION

I conclude this talk by returning to its beginning: CAS−based tasks require construction and interpretation of signs with different rules for production and interpretation as compared to paper-and-pencil math, although the underlying meaning structure of mathematics is usually the same in both cases. This meaning structure is both reflected by and constituted through the activities with signs, be they paper-and-pencil or CAS−based. In particular, the construction of appropriate CAS−based signs is often problematic for students, and the level of guidance in this regard is a challenge for the teacher. At the same time, complexities around the construction of signs may open up new possibilities for task design. Furthermore the teacher needs to be aware how the provision or withholding of certain information in a task statement may enable or inhibit experimentation or interpretation. This is surely an ongoing challenge and requires further research.

Hopefully this talk has suggested new ways of looking at the structure of tasks and demonstrated how a small change in the task design can effect a large change in the sort of learning that takes place.

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