# ENHANCING CONCEPTUAL INSIGHT USING A CAS

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Abstract. When a graduate solves a mathematical problem using technology, he/she generally uses a single one-step "high level" command of a readily available CAS to obtain the solution immediately. This should not happen during his/her training. In a Mathematics course, the educator should decompose the solution process into elementary steps and reinforce each step by a "low level" usage of a CAS. This approach is absolutely essential in order to provide the student the conceptual insight into the solution process. We illustrate this with a few mathematical topics.

## 1. Introduction

> "*The use of technology in teaching college mathematics is a pressing issue, currently with more questions than answers*" ([Murphy and al. 2002]).

Practicing professionals are commonly used to thinking that a single algorithm is sufficient to solve a given mathematical problem and accept the CAS output as definitely true and useful forgetting the limitations, the occurence of wrong entries or problems with the underlying algorithms of the particular CAS. This is a very dangerous practice and should definitely be avoided during the professionals' training. If the student obtains an answer from a CAS and uses it as it is, without gaining any insight into the mathematical processes involved and probably not realizing the existence of other (superior) methods or algorithms different from the standard ones, he/she cannot check the correctness of the solution and certainly has absolutely no chance to invent his/her own methods.

Among educators there exist two different points of view about the usage of software (see [Lagrange 2000]): the software can be viewed either as a tool making the technical part of the work easier, or as an experimental environment helping the student to achieve a better understanding of the concepts.

Whether we use a CAS as a tool or as an experimental environment, the main goal of an educator will still remain to lead his/her students toward a better understanding of the conceptual part of mathematics. As M. Artigue points out in [Artigue 2001]:

> "*Techniques do not have only a pragmatic value which permits them to produce results, they have an epistemological value, as they are partly made up of the understanding of the objects, and are also a source of new questions.*"

When teaching a Mathematics course, the educator has to make an important choice:

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1. Teach the course in a classical way, i.e. only with blackboard and chalk; solving exercises is time-consuming and, in order to show various situations, the educator has to choose quite simple examples.
2. Teach the course using technological tools; this way offers an opportunity to develop more complicated examples than in the blackboard and chalk teaching, examples which would be impossible to solve without software. With this choice, the educator has further to decide still whether to teach the course using:
   (a) a textbook designed to treat each topic coupled with a single "high-level" command of some specific software;
   (b) a series of "low-level" commands of some software, i.e. simple algebraic commands existing in practically every package or symbolic calculator, to explain the solution process.

Pedagogically, we believe that the choice (2b) is imperative and absolutely essential in order to ensure a better insight into the mathematics involved with the added advantages of challenging the students' faculty and developing their creativity and originality. The usage of "high-level" commands should be postponed to a further step of the students' cognitive process, only after the necessary theory has been acquired. Such a cognitive process enables the student to make original decisions, such as "what is the best technique to follow after each step". As Herwaarden and Gielen say (see [Herwaarden and Gielen]):

> "*... students often lack conceptual insight while using a computer algebra environment. A reason for this seems to be that they don't incorporate the computer techniques into their mental approach of mathematics. Because the students have learnt mathematics using paper-and-pencil methods and their mathematical way of thinking has developed in close relation with these methods, one can suppose that a good internalisation of computer techniques can be reached by an appropriate link with paper-and-pencil methods.*"

At a more advanced level, once the students have gained this conceptual insight, they will be able to deal with more difficult assignments, where paper-and-pencil methods either are too heavy or demand too complicated techniques for the task to be performed in a reasonable amount of time. In such a situation, the technical computing skills together with the conceptual understanding already acquired can justify a direct usage of a CAS.

Using a few mathematical topics as examples we describe how a student gets acquainted with each topic by decomposing the solution process into elementary steps, then using a CAS for the computations involved in each step.

## 2. Laplace transform

The solution of a linear ODE using Laplace Transforms is one of the key techniques in the modern approach to the *analysis* and design of engineering systems. The process for obtaining the solution $f(t)$ of a linear ODE with Laplace Transforms should be decomposed into three basic steps:

a. Transform the linear *differential equation* from the "time $t$ domain" into an *algebraic equation* in the "frequency $s$ domain" by taking the Laplace Transform of both sides, generally using a table of transforms, together with as many as possible properties of the transform, such as linearity, transform of derivatives, translation properties, convolution, etc. These formulae are generally also included into the tables, so that students can solve exercises without knowing how these formulae were derived. This in itself surely causes a lack of understanding of the mathematical topic, not to mention the understanding of the basic underlying physical concept, and makes the student act mechanically.

b. Solve the algebraic equation for the transform $F(s) = \mathcal{L}(f(t))$; the function $F(s)$ is generally a combination of rational functions in the variable $s$ and/or of products of such functions by exponentials of the form $e^{as}$. This task is purely technical and software can help perform it.
c. Compute the inverse Laplace Transform $f(t) = \mathcal{L}^{-1}(F(s))$ of the transform which has just been found.

Most CAS contain both a single “high-level” command to find the solution, as well as all the built-in ”low-level” commands to perform the above three steps of the solution process.

Recall that the computation of the *Laplace transform* of $f(t)$ is the symbolic evaluation of the improper integral:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st}\, f(t)\, dt \tag{1}$$

for values of $s$ such that the improper integral is convergent.

The evaluation of the improper integral involves the computation of:

(i) a definite integral with a parameter as its upper boundary,
(ii) a limit at infinity.

A CAS has generally built-in commands for computing the Laplace transform of a function; however, this does not help the student to understand the mathematical meaning of the computations. Moreover, it is not always possible to use directly the result displayed by the computer.

For example, the hand computation of the Laplace transform of $f(t) = e^{-t}$ readily yields the correct result:

$$F(s) = \lim_{\lambda \to +\infty} \int_0^{\lambda} e^{-(s+1)t}\, dt = \frac{1}{s+1},$$

whereas the use of the command **LAPLACE** $(f(t), t, s)$ of a certain CAS returned

$$\frac{1}{\mathrm{LN}(e) + s} - \frac{1}{\infty^{\mathrm{LN}(e)+s}\,(\mathrm{LN}(e) + s)}.$$

This display is problematic for three reasons:

1. $\ln e$ has not been evaluated;
2. it includes the $\infty$ (not representing any number) as a symbol in an algebraic formula;
3. it avoids the computation of the limit at infinity.

The designers of this software have recently realized the inadequacy of this display, and have upgraded it to

$$\frac{1}{s+1} - \frac{1}{\infty^{s+1}\,(1+s)}$$

but this is clearly only a partial remedy to the problem. In pencil-and-paper work, the educator would never accept such notations (maybe the programers did not pay attention to the problem?).

The resulting algebraic equation for the unknown $F(s)$ can generally (but not always) be solved using a single command of a CAS, but the solution displayed is often not suitable for computing the inverse Laplace transform. For example, the solution

$$F(s) = \frac{4s+3}{s^2+9}.$$

does not appear in any table of Laplace transforms, hence one further needs to express this rational function in the invertible form

$$F(s) = \frac{4s+3}{s^2+9} = 4\frac{s}{s^2+9} + \frac{3}{s^2+9},$$

to yield

$$f(t) = 4\cos 3t + \sin 3t.$$

Even when a CAS computes this inverse Laplace transform, we (certainly the students) do not know whether it uses this kind of decomposition or another way.

For more complicated functions, there is an additional need to first decompose the rational fraction into partial fractions, and then express the result into a suitable form for inversion. Besides enabling the inversion, the partial fraction decomposition offers the student the insight into the type of solution he/she may expect (e.g. sinusoidal, transients, etc.). This insight is often more important than the solution itself.

A similar situation arises in more complicated problems where the input (or forcing) function on the right-hand side of the differential equation has discontinuities. In these problems, $F(s)$ contains products of rational function with exponentials. The presence of these exponentials for every discontinuity, makes computations a little harder, at least from the student's point of view. Here the software can make the fear level of the student lower, but not every CAS has a built-in Heaviside function, and some (time-consuming) hand computations still remain. Again the knowledge that the discontinuities in the time domain are encoded as products with exponentials in the frequency domain is very important to gain the insight into the type of solution expected.

Let us focus on another, perhaps more important, aspect of this topic. In addition to being a very powerful unified and useful solution technique for solving linear differential equations, Laplace Transforms also provide the engineer with a greater insight into system behaviour. More often than not an engineer is interested not only in system analysis but also in system *synthesis* or design. Consequently, an engineer's objective in studying a system's response to specific inputs is frequently to learn more about the system with a view to improving or controlling it, so that it satisfies certain specifications. This is an other area where the use of Laplace Transform is very attractive, relatively easy to apply and widely used in practice. Without a knowledge of the underlying mathematical theory and properties of Laplace Transforms, this would not be possible. The use of "high-level" commands would totally avoid the acquisition of this knowledge.

The following short compendium of student feedback confirms our claims above and reflect how the students' learning experience was improved, both from a stimulating and conceptual point of view.

> *"Laplace transforms is a great topic. For the very first time I realized how much easier it is to work in a new (frequency) domain!"*

> *"I just can't believe it! I worked out an inverse transform in four different ways. Not bad for an engineering student!!"*
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> *"The treatment of Laplace transforms in the maths course, particularly that of transfer functions, helped me greatly in my control systems design course. The block diagram representation makes it so much easier to visualize and comprehend how the various parts of the system interact."*
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> *"In the mathematics course we learned of the equivalence of algebraic multiplication in the frequency domain and convolution (star product) in the time domain. I always prefer a simple multiplication to a convolution, no wonder frequency design techniques are so popular in engineering design."*
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> *"It took me a while to understand the equivalence between the transfer function in the frequency domain and the impulse response in the time domain of a LTI-system. This understanding makes design a breeze."*
>
> *"Fourier, Laplace and* $z-$*transforms are basically the same. If you know Laplace, the others are easy. Signal processing is simply their application."*

## 3. Ordinary Differential Equations

Consider the following ODE with initial condition:

$$\begin{cases} xy'' + 2y' - y = 0 \\ y(0) = 1; y'(0) = -1 \end{cases} \tag{2}$$

It is not rare to see a student trying first to solve the given homogeneous differential equation, regardless to the initial condition. Some Computer Algebra Systems have an algorithm for second order linear ODEs, so he/she can even do the job using a computer. Nevertheless, all this work has no value; hopefully he/she will discover this when introducing the initial value, but this is not sure. The theorem which ensures existence and unicity of a solution demands that the coefficient of $y'$ will be equal to 1 and that all the other coefficients will be continuous functions. We obtain this by dividing Equation (2) by $x$; the point 0 does not belong to any continuity interval of the coefficients. The usage without care of the CAS lead to nowhere. As says T. Etchells (see [Etchells 1993], quoted by [Mackie 2002]):

> "A very didactic problem with CASs is that they can produce meaningless expressions. Students are punching keys and performing operations on expressions that have no meaning; they are producing mathematical nonsense."

In such a situation, the educator must react and propose a pedagogical activity "ab absurdo". Anyway, we do not think that this is the right way to build a Computer Assisted Course in Mathematics.

## 4. Parametric systems of linear equations

Consider an $n \times n-$matrix $A$ whose entries are functions of a real parameter $m$ and an $n-$dimensional vector $B$. Solving the system $AX = B$ is a classical question, appearing in any course in elementary Linear Algebra.

For the non-parametric case, students use either the Gauss-Jordan method (putting the matrix in echelon form) or the Cramer method, with determinants. The necessary commands are implemented in every symbolic calculator and every CAS: for example the **rref** command in the TI-89 and TI-92 and Derive's **ROW_REDUCE** command put a given matrix in echelon form. Solving a system of linear equations with these commands yields an immediate answer. Sometimes there exist another command like **solve** which provides an immediate answer, but the student cannot know which algorithm has been used. This does not give a profound insight into the solution process and the CAS is used as a blackbox.

Cramer's method uses the **det** command and the student must decompose his/her work into elementary steps, computing $n + 1$ determinants and then dividing out. This way gives a better conceptual insight, at least for one method, than the previous one.

For a system of $n$ equations with $m$ unknowns ($m \neq n$), using Gauss-Jordan's reduction should be better. In order to obtain an insight into the algorithm, instead of using the high-level commands (**rref**, or **ROW_REDUCE**), the educator should introduce the commands corresponding to elementary operations on the rows of the matrix. This will have a great influence later, when solving numerically systems of equations: when stability of the solutions is an issue, the choice of the pivots is of the utmost importance. Using commands for performing elementary operations contributes to the students' understanding of the involved phenomenon.

In the parametric case, the situation is more intricated. The matrix of the system can still be put in echelon form, but only the general case is considered and the special values of the parameter(s) are hidden. The student can be convinced that he/she actually solved the problem, when he/she did not.

In order to built an activity for solving a parametric system of linear equations, we should propose the following canvas:

- (i) Compute the determinant of the principal matrix of the system;
- (ii) Solve the algebraic equation, looking for zeros of the determinant;
- (iii) Still using determinants, solve the system for the non singular case.
- (iv) Substitute one value for which the determinant vanishes into the augmented matrix of the system;
- (v) Row reduce this matrix;
- (vi) Check whether the system has solutions or not. If there are solutions, compute them.
- (vii) Iterate the last three steps as many times as needed.

Be aware to the fact that this canvas uses only low level commands of the CAS. By this means, the teacher leads his students to a more profound insight into every step of the solution process. We should emphasize that some steps can be decomposed into more elementary actions. For example, in order to row reduce the matrix, don't use a single command like **rref**. Instead, the CAS enables you to perform elementary operations on the rows of the matrix.

We should emphasize another point: students ask sometimes questions like "what is it good for?" or "why do I need to learn another method? I already know one and it works!". The case of parametric systems of linear equations gives a clear answer: the computerized solution of such a system must involve more than one method. For more details and examples, see [DP 2000].

## 5. Conclusions

> *"The end of our second year Electrical Engineering is coming. Throughout it we've found that Maths has played an absolutely important role. Whatever you've taught is very precise and a great thing is that you don't mind helping to explain some problems from the other subjects."*

As this excerpt which appeared in the campus student magazine illustrates, our claim is also true for other mathematical topics and techniques taught in Engineering Mathematics courses.

During his/her learning cursus the engineering student must be provided with a perfect balance between mathematical techniques and conceptual understanding.

The conceptual understanding of the mathematical techniques, at every step of the solution process, is absolutely necessary. Without the ability to think conceptually in mathematical terms, the engineering students hence the future engineer will be unable to tackle engineering problems. With the use of "high-level" commands without a proper understanding and internalizing of the underlying mathematical and physical principles involved, the engineer would simply become a spectator of modern technology with absolutely no innovative power. We wish to conclude with Mackie's conclusion in [Mackie 2002]:

> *"The power of computer algebra goes beyond routine computation. It has the potential to facilitate an active approach to learning, allowing students to become involved in discovery and constructing their own knowledge, thus developing conceptual understanding and a deeper approach to learning."*

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