FORUM 1/PRESENTATION

A didactic approach of the use of Computer Algebra Systems to learn mathematics

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The themes of the workshop invite presenters to look at the links between “theoretical” work in mathematics education and classroom practice, with particular respect to the role of CAS and to highlight the role of *explicitness* and *expressiveness* in CAS use by students.

In this paper I will first consider how a theoretical didactic approach can help to look at the use of CAS in the learning of mathematics. Studying the use of DERIVE in French classrooms, I observed that the approach that teachers wanted to develop with CAS was often too conceptual and neglected the role of the technical work in the learning. So there was a need to consider what the technical work is, both with and without CAS and how it is linked with students' understanding of mathematics. I will show what theoretical elaboration helped to conceptualise these links and to highlight the role of CAS and non-CAS techniques. I will then consider expressiveness, an area which has not been extensively studied, by examining a sequence of classroom activities in pre-calculus.

Background

My research domain is didactics. In France “didactique" began some 25 years ago and, over the years, it has interacted with the mainstream of Mathematics Education. It has, however, its own conceptualisation (see Kilpatrick, 1994, Sierpinska, 1995). This paper aims to highlight a contribution to the reflection on CAS use.

My involvement in didactical work with CAS started in 1994. The French Ministry of Education was then experimenting with the use of DERIVE in the upper secondary level (see Hirlimann, 1996). A working group of teachers had to explore and experiment with the potential of CAS for mathematics teaching and learning. A team of "Didacticians" (DIDIREM) was invited to join the group to study this experiment. In this study, we looked at the use of DERIVE in France, investigating the representations of the teachers on the potential of CAS and comparing these representations and the reality of the classroom observations and of the students’ attitudes (Artigue, 1997 ; Lagrange, 1996). After this, in 1996, we began a new study to assess the relevance of the TI-92 symbolic calculator for the learning of pre-calculus in the science stream of French schools. In contrast to the DERIVE study we did not attempt a survey of classrooms practices. Working with teachers in two classes during two years, we built a pilot project for the teaching of pre-calculus.

The role of techniques using CAS

In the teaching of mathematics, and especially in algebra, a tension existed between manipulation and understanding. "Teachers (in the USA) even (in 1890) were opposed to what they saw as an overemphasis on manipulative skills and were calling for a meaningful treatment of algebra that would bring about more understanding" (Rachlin, 1989). In the past 15 years, the idea emerged strongly that universal access to the new technology would "enable us to modify our skill-dominated conception of school algebra and rebalance it in favour of objectives related to understanding and problem solving" (Kieran and Wagner, 1989). Reservation about this agenda was that it would require "the mathematics education community (to) deal(s) with the four factors of teachers, textbooks, evaluation and articulation between (...) coursework" (ibid.), a deal not yet achieved, to my mind.

To many authors, CAS is an appropriate technology for this "new balance" because it is not limited by the approximate treatment of numbers nor by the necessity of programming. Therefore it is not surprising that in the research papers on CAS, authors generally study how CAS might help to "set a new balance between skills and understanding', or to "resequence skills and understanding" (see Mayes, 1997, for a review).

In France, the picture is not exactly similar. First, research studies on the teaching of algebra opened up a didactic elaboration stressing the interaction between technical and theoretical dimensions of the mathematical activity, rather than opposing them.

Second, with the French centralised curriculum, research had to focus on the integration of CAS into this curriculum rather than to assess improvements of students' conceptual reflection in small scale projects. For this reason our interest was in the changes produced by the introduction of CAS in the everyday teaching and learning of mathematics and in the search for conditions in order that these changes bring satisfactory effects. As CAS has a deep impact on the procedures of doing mathematics, the study of these changes was a "window" opened on the role of techniques in the learning of mathematics.

I will first outline the above mentioned didactic elaboration, then look through the window in the light of this elaboration.

A didactic approach of the role of techniques

Researchers in France were concerned that, with the evidence of pupils' difficulties, and the emphasise put on problem solving and application, algebra could disappear of school mathematics. While the manipulation of symbols was seen by many as meaningless, and since calculators could provide means to solve numerically the problems, why should teachers continue to ask pupils for algebraic resolution, a task in which a number of them fail? The researchers first emphasised the power of algebra and its potential place as a foundation of school mathematics (Mercier, 1996). Then they had to reflect on the reasons of this poor position of algebra in the reality of pupils and students' mathematical activity. They stressed that because of successive reforms, algebra existed in schools just as scattered techniques. First, no more tasks existed that these techniques could help to achieve, because less emphasis was put on proofs. Then, there was no reflection on these techniques which could make them visible mathematical entities.

From these observations they stressed that a mathematical topic should exist in school as a consistent set of tasks, techniques and conceptual reflection. With this set, student at first see the tasks as problems. Progressively they acquire the means to achieve them, and then they become skilled. That is how they acquire techniques in a topic. Furthermore, in the teaching and learning situations, the students and the teachers are not interested in simply acquiring and applying a set of techniques. They want to talk of them, and therefore they develop a specific language. Then they can use this language to question the consistency and the limits of the techniques. In this way, they reach the theoretical level in the topic. Chevallard (1997, p.37) names "praxeologies" these sets, as they involve "praxis" and "logos".

Techniques in the CAS context

In the DERIVE research study, our team questioned teachers using CAS on their view of the support that it brings to the teaching and learning of mathematics. We found a common assumption: CAS lightens the technical work in doing mathematics, and then students will focus on application or understanding

On the other hand, our survey of the French classrooms in the DERIVE experiment showed neither a clear lightening in the technical aspects of the work nor a definite enhancement of pupils’ conceptual reflection (Lagrange, 1996). Technical difficulties in the use of CAS replaced the usual difficulties that pupils encountered in paper/pencil calculations. Easier calculation did not automatically enhance students’ reflection and understanding.

My idea is that the teachers did not perceive the need for a consistent praxeology: they wanted to jump directly from the tasks to the conceptual reflection, without seeing that this reflection operates on techniques, and not on tasks. So it appeared that many students did not consider problem solving using computer algebra as a convincing support of their understanding of mathematics, even when they liked it. They felt that their understanding developed from the techniques that they built in the ordinary context, and solving problem with CAS seemed to them very apart from these techniques. Moreover, the teachers were not used to think about the techniques that are necessary to achieve a given task. So they were not able to see that techniques specific to the use of symbolic computation exist, that they are not obvious and that they might be also a topic for a reflection.

CAS specific techniques

To explain this, I consider the problem of the factorisation of $x^n - 1$ as developed by Mounier and Aldon, (1996). These teachers wished that tenth grade students find general factorisations of this polynomial and regretted that, in the paper/pencil context, students were restricted in their investigation by the difficulties of the calculations and notations. They tried to use DERIVE to liberate students "from the technical aspects of computing by hands" and then "to keep sight of the main goal". Observing a set of outputs from DERIVE’s **Factor Rational** command, several students suggested that a general factorisation should be $(x-1)(x^{n-1} + x^{n-2} + ... + x + 1)$. But others objected that this is true only for odd values of *n*. They had observed DERIVE’s factorisation for $n = 4$ and $n = 6$, and they said that with *n* even, the factorisation should be $(x-1)(x+1)(x^{n-2} + x^{n-4} + ... + x^2 + 1)$. Others found that the suggested general factorisation is not true for every odd *n*, because $x^9 - 1 = (x-1)(x^2 + x + 1)(x^6 + x^3 + 1)$. Then, all the class agreed that they should search for other general factorisations and they did not find them[1].

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[1] They could discover techniques to get factorisations in a number of situations (for instance, when *n* is prime, or a product of two primes). However, a satisfactory ‘theory’ issuing from these techniques (the theory of cyclotomic polynomials) is beyond the reach of tenth grade students.

In my analysis, students encountered a conceptual difficulty to grasp this: for a given n, several factorisations may exist, but only one is "general" (i.e. true for every integer n). Using CAS, students met this difficulty, because the DERIVE's **Factor Rational** command returns the most factorised form of a polynomial and this form is "general" only for n prime. This situation is interesting as students will have learned a lot about algebra if they can say: "Yes DERIVE gave us factorised forms of given polynomials which are not the general factorisation, but the general factorisation is still true, because we can get it by expanding parts in the factorised forms". I stress, to be able to say that, students should know means to select and expand a part in these forms. A mathematician may think that this "technique" is obvious, because (s)he is able to recognise complete and incomplete factorisations and (s)he conceives that CAS provides the means to pass from one to another. By contrast, students were not easily able to select a part in an expression and to expand it. In this difficulty, their poor ability to manage factors in DERIVE could not be separated from their (mis)understanding of the concept of factorisation.

We observed that teachers in the DERIVE experiment were often reluctant to give time to techniques of management of expressions in DERIVE. Since it was difficult to work regularly in a computer room, most of them did only a few sessions with DERIVE. In this context, they saw little use for DERIVE's techniques and tried (often unsuccessfully) to focus on conceptual issues. By contrast, experienced teachers and researchers, had success when they integrated DERIVE's techniques in the usual classroom. For instance Mounier's and Aldon's students had access to DERIVE on laptops in the classroom and for personal work. These teachers set the factorisation of $x^n - 1$ as "a long term problem": according to them, students did learn DERIVE as a new tool *and* changed their image of the concept of factorisation. These teachers recognised the need for building techniques for using DERIVE and the role of these techniques in the understanding of algebra.

The role of non CAS techniques

I said that the aim of French research studies was the search for conditions in order that the changes introduced with CAS bring satisfactory effects. In the light of the above didactic elaboration, I would say that a goal should be to produce new praxeologies. Obviously, it is not easy. With an example of a task, I want to show how, beside CAS techniques, "ordinary techniques" are necessarily involved in praxeologies and then might not be avoided.

This task was given to a 12th grade class where every student had a TI-92 (Trouche et al., 1998). The "challenge" was: for every integer n find the n^{th} order derivative of $(x^2 + x + 1)e^x$.

Using the TI-92, this task might be just recognising a pattern[2]. Proving a general formula as well could be done using the TI-92 without knowledge in calculus.

On the other hand, this task is worth a deeper reflection. In (Trouche, ibid.) two students[3] wrote: "Using the TI-92, we recognised a pattern and proved it. We have now to look again at this exercise. Actually, we search for the derivative of a product of two functions u and v, with $u(x) = e^x$ and $v(x) = x^2 + x + 1$. Every derivative of u is u, the first derivative of v is $v'(x) = 2x + 1$, the second is $v''(x) = 2$ and the other derivatives of v are zero. From this, we

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[2] Recognising this pattern requires specific abilities with the TI-92, because the pattern does not appear clearly in the TI-92's simplified derivatives. They are related to algebra and not to calculus.

[3] This booklet was done from the reports of students of Luc Trouche after they solved a number of "challenges".

calculate the first, second and third derivative of *uv*, then we use the formula of Leibnitz and find $(uv)^{(n)} = uv + nuv' + \frac{n(n-1)}{2}uv''$. From this, we find again the expression for the nth order derivative of $(x^2 + x + 1)e^x$."

In the first part of their work these students elaborated a technique that they could apply to every product of the exponential function and of a quadratic polynomial: try a number of derivations, guess a pattern, and use the TI-92 to prove it. Why did they engage in this additional reflection? Because, as mathematics students, they wanted to know why this technique works and to link it with a more general knowledge in calculus.

I stress that, in this conceptual reflection on a TI-92 technique, students had to refer to the usual techniques of differentiation: seeing a function as a product, differentiating polynomials and exponential, differentiating products. More generally, many tasks especially in calculus, are done easily with CAS, because results like limits, derivative and integrals are simple to get, but they have a value as a mathematical activity only if students are able to link these results with techniques of symbolic transformations. Moreover, students cannot really understand these techniques in calculus without enough algebraic knowledge (about the algebraic knowledge required to understand symbolic differentiation, see Pozzi, 1994).

So, an adequate use of CAS should not try to rid the learning of techniques. It should, in contrast take advantage of the availability of new techniques to foster students' conceptual reflection.

The expressive power of CAS?

I come now to the role of the language of CAS, a question of which, in France, we have only just scratched the surface.

My starting reflection is that CAS is not the first technology in the learning of mathematics to integrate a language. In micro-worlds (Hoyles and Noss, 1996, p. 68) "the central technical actors are computational objects" and "it is the language, the program, which allows the most obvious link between computational and mathematical discourses". Hoyles and Noss (ibid. ch. 4) give convincing examples of the "expressing power" that this language gives students when working with the calculator, and also in classroom interactions. Thus, seeing the use of CAS as acting in a micro-world is tempting[4]. Expressions in CAS would be the concrete form of abstract algebraic and analytic objects. However, there are also clear differences. Objects in a micro-world evoke "already known" entities (Papert's turtle is the most famous) and are built and handled through a programming language while, in CAS, basic objects are algebraic expressions, and the common use is the simplification of single expressions. Nevertheless, I will try to search for similar contributions in the experiment of TI-92 use.

Expressing enactive ideas in calculus.

To consider the expressing power of CAS, I will look at CAS as a means to express an enactive differential knowledge using computable representations.

I take the idea of enactive knowledge and of computable representations from Tall (1996) who stresses on the "spectrum of possible approaches to the calculus". The first approach, "enactive" uses the common differential knowledge: for instance, people generally have a

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[4] Kent and al. (1998) look at the design of activities using Mathematica as the creation of micro-worlds.

sense of motion and velocity from their everyday experience. The second level, "theoretical', includes computable representations of concepts. At this level, students work on algebraic expressions of functions, on graphs, on numerical values.....

It seems important that students in the initial stages of learning calculus link their new theoretical representations of functions and their enactive differential knowledge. For instance they have to understand that the geometric property "translating a curve along the x-axis doesn't change its slope" has to do with a property of the derivative: f being a function defined near 0, and differentiable at this point, the function $x \to f(x-1)$ is defined near 1 and differentiable at this point, and its derivative at this point is $f'(0)$.

In the TI-92 experiment, we chose to stress on this link on several occasions:

- first, when introducing the concept of derivative, we started from the students' geometrical experience of tangent lines to curves,
- later in the year, we asked students to express the velocity of a car, from a curve representing its motion, and then to find that the derivative is the appropriate means to do that,
- in application problems, students had to model "real world" situations, expressing these situations as constraints for an unknown function (see the "pipe problem" in Lagrange, to appear).

Hypothesis

Stressing this link is obviously not new. For instance, in many approaches of the derivative, students have to look at the graph and to guess the slope of the tangent line, then the teacher shows how this number might be calculated through the limit of the slope of a secant line. But in an ordinary context, this link is weak: students often forget the enactive introduction and see calculus just as a manipulative activity. Our hypothesis was that CAS could bring deeper links between the enactive knowledge and the computable representations.

- First, this link has not be just "using a prior knowledge". Since this knowledge is often too poor to really be linked with calculus, introducing theoretical calculus has also to change it. So the multiple representations of CAS may be used to look at puzzling phenomena and to question students' prior knowledge.
- Second, students have to build this link through action. CAS provides means to handle expressions more easily than using paper and pencil, and thus, students may act on a larger set of functions with possibly more complex expressions. Moreover, with CAS, students might look differently at the expressions. The paper/pencil transformations are "rule oriented" when CAS makes easier to recall the goal of the transformations.

I will study the introduction of the derivative as an example of how the above hypothesis on the potential of CAS for a deeper link between enactive knowledge and theoretical calculus may work. I will first specify the hypothesis for this introduction, then I will report on a classroom experiment.

Students' prior enactive knowledge of the tangency of a curve and of a straight line is geometric. Using Castela's (1995) work, I will show how this knowledge might be questioned in an approach of the derivative. High school students see the tangent line through their experience of the intersection of a circle and a straight line. In this experience, tangency is a position, half-way between no intersection and two intersecting points. So, students admit that a circle and a tangent line have one point in common. On the other hand, they realise that the line and the circle are "locally very close". Generally, their representations do not admit the possibility that the tangent goes "through" the curve.

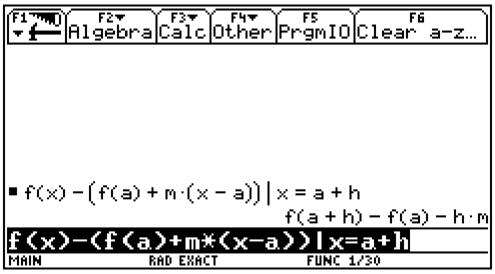
Students' prior representations are not consistent and depend on the curve. For instance, when the curve is a straight line, students prefer the "one intersecting point" representation: they think that every intersecting straight line is tangent, except the line itself. When the curve is the graph of a cubic function, the "curve on one side of the tangent line" representation makes them difficult to consider the tangent line at the centre of symmetry. The "locally very close" representation is interesting, because it is a "differential" representation. It is also not explicit, because it is easy to see that a curve and a line are locally close, but it is difficult to visualise whether a line is "closer to the curve" than another.

This limitation of the visualisation is a motivation to search for an analytic expression of this "local closeness". CAS makes the visualisation easy and may show this limitation: zooming cannot help to see a difference between two "locally close" lines. Furthermore, since the expressions in the graphical and in the algebraic windows are the same, CAS helps to shift from a graphical view of a phenomenon to an analytic expression. For these two reasons, CAS is expected to help students to question their prior knowledge, and to switch to an analytic view. Then this analytic view of the "local closeness" of a line and a curve will help to look again to other representations like "not through the curve".

According to above second hypothesis, CAS helps also to act more easily on expressions and to give another signification to the algebraic transformations. This is how it is expected to work in the search for an analytic expression of the "local closeness".

f being a function, $y = f(a) + m(x - a)$ is the equation of a line of slope *m* cutting the curve at the point of abscissa *a*. Searching for the values of *m* to make the line and the curve 'close', students have to deal with the distance between a point M on the curve and a point P on the line such that $x_M = x_P = \underline{a+h}$ (*h* being a "small" number). This distance expresses as the difference of the expression of the function, and of the right side of the equation of the line. *f* being a given function, transforming this difference may be done either through paper and pencil, or through the use of adequate commands of CAS. Working with paper and pencil, a student has to recall the rules of transformation, for instance if *f* is cubic, (s)he has to consider the expansion of $(a+h)^3$, or to cleverly transform $(a+h)^3 - a^3$. Meanwhile it is difficult for him (her) to remember both the starting point and the goal: the transformation of the difference to show the behaviour of the line and of the curve.

An hypothesis is that CAS might help to focus on the goal As shown in the screens below, using a TI-92, a student is able to enter an expression of the difference, then to search for a useful view of the behaviour of this difference near 0. When *f* is a polynomial, the expansion into a polynomial of the variable *h* is the appropriate choice. With a non polynomial function, algebraic transformations do not work, and a limit has to be considered.



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■ f(x) − (f(a) + m·(x − a)) | x = a + h
                              f(a + h) − f(a) − h·m
f(x)-(f(a)+m*(x-a))|x=a+h
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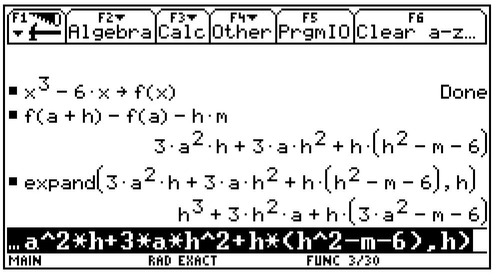

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■ x^3 − 6·x → f(x)                                 Done
■ f(a + h) − f(a) − h·m
                  3·a^2·h + 3·a·h^2 + h·(h^2 − m − 6)
■ expand(3·a^2·h + 3·a·h^2 + h·(h^2 − m − 6), h)
                    h^3 + 3·h^2·a + h·(3·a^2 − m − 6)
…a^2*h+3*a*h^2+h*(h^2-m-6),h)
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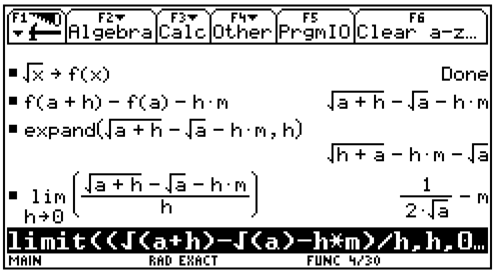

- $\sqrt{x} \to f(x)$ Done
- $f(a+h)-f(a)-h\cdot m$ $\sqrt{a+h}-\sqrt{a}-h\cdot m$
- $\text{expand}(\sqrt{a+h}-\sqrt{a}-h\cdot m,h)$ $\sqrt{h+a}-h\cdot m-\sqrt{a}$
- $\lim_{h\to 0}\left(\frac{\sqrt{a+h}-\sqrt{a}-h\cdot m}{h}\right)$ $\frac{1}{2\cdot\sqrt{a}}-m$

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limit((√(a+h)-√(a)-h*m)/h,h,0…
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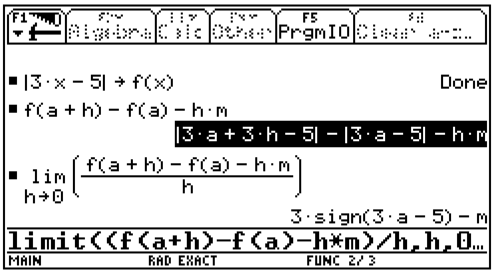

- $|3\cdot x-5| \to f(x)$ Done
- $f(a+h)-f(a)-h\cdot m$ $|3\cdot a+3\cdot h-5|-|3\cdot a-5|-h\cdot m$
- $\lim_{h\to 0}\left(\frac{f(a+h)-f(a)-h\cdot m}{h}\right)$ $3\cdot \text{sign}(3\cdot a-5)-m$

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limit((f(a+h)-f(a)-h*m)/h,h,0…
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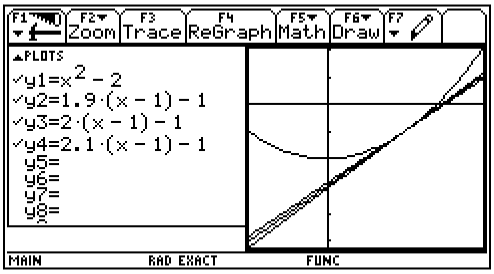
Experimenting

We (the DIDIREM team) experimented with this approach to the derivative. We chose to start from the study of a quadratic function defined by $f(x)=x^2-2$ at a given point, then to continue with a cubic function at different points and finally to study non polynomial functions at an arbitrary point.

This is how the lesson went on. The teacher introduced the parabola representing the function *f* defined by $f(x)=x^2-2$ and A the point (1 , -1). Using a Cabri-geometre animation, she displayed the parabola and moved a straight line passing through A. During this animation the slope of this line was also displayed. She asked students to search for the value of the slope which gives the "best fit" of the graph.

Students guessed easily that the slope of the line of best fit is 2. Then the teacher moved the line a little and stressed that the lines of slopes 1.9 and 2.1 also fit well. Students entered the function and the equations of the three lines in the base of the TI-92 and switched to the Graph window. They tried several zooming and recognised that it is not possible to distinguish the "fitness" of the three lines.



✓$y1=x^2-2$
✓$y2=1.9\cdot(x-1)-1$
✓$y3=2\cdot(x-1)-1$
✓$y4=2.1\cdot(x-1)-1$
y5=
y6=
y7=
y8=

The teacher offered to consider the distances between the curve, near A, and the lines. The first two columns of the table below display the expressions of this distance that students found for the three lines.

Students obtained these formulations by hand. They drew a sketch of the curve and of a line, put the co-ordinates of the points with explicit more or less simplified formulations (i.e. $(1+h)^2-1$ and $1.9(1+h-1)-1$) and simplified the difference from these formulations.

Students observed that the distance between the curve and the line of slope 2 simplifies into a one term expression, when the two others simplified in two terms expressions. Students were interested by the reason why. The discussion with the teacher highlighted the expansion of $(1+h)^2$ and the cancellation of the terms of degree one.

The discussion on the "best fit" was then easy. Students were aware that two functions may have the same limit for *h* approaching zero, but not the same local behaviour and where thus able to recognise that h^2 "tends faster toward zero".

Furthermore, it appeared in the discussion that considering the limit of the quotient of the distance over h is handier and a column was added .

Slope	Distance	Limit of quotient
1,9	$h^2 - \frac{h}{10}$	$-\frac{1}{10}$
2	h^2	0
2,1	$h^2 + \frac{h}{10}$	$\frac{1}{10}$

Actually, students found that the line of slope 2 fits better than the line of slope 1.9 or 2.1 and not that it fits better than any other line. The teacher emphasised this and recalled the equation of an arbitrary straight line passing through A. The expression was then parametric and the students added the following row in the table.

Slope	Distance	Limit of quotient
m	$h^2 - h\,(-m+2)$	$-m+2$

They found that the line would fit the best when this expression simplifies to zero.

In the subsequent session, the teacher asked to do the same work with the cubic function $x \mapsto f(x) = \frac{x^3 - 6x}{4}$ at the point of abscissa 2. She asked again to find the "best fit" among three lines, and then the "best fit" among all lines. This time, the students used their calculators. They adapted by themselves the research of the previous session, concentrating on the goal.

So they found that the slope of "best fit" was 1.5. When they drew the curve and the line, they were surprised because the tangent line had another intersecting point, a situation that they did not encounter with circles and parabolas. Then they had to do the same task for the point of abscissa 0 and get another puzzling phenomenon because the tangent "goes through the curve".

Starting the third session, the teacher recalled what the students had done. She stressed that searching for the "best fit" at a point of abscissa a was to search for a slope m such that

$$\lim_{h\to 0}\left(\frac{f(a+h)-f(a)-h\cdot m}{h}\right)=0$$

and that this slope might be computed as $\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$.

So the students knew that they were actually computing derivatives and they had to do that for the following functions: $x \mapsto \frac{1}{x}$ $x \mapsto |3x-5|$ $x \mapsto |x^2-4x+3|$ $x \mapsto x\sin\left(\frac{1}{x}\right)$.

The work was carried out over two sessions, mainly using the calculator. Students generally defined the function as f, then computed the limit directly, from the above expression. Some of them had difficulties in expressing this limit on their calculator because of the two variables (a and h). We observed also that students often came back to the quotient $\frac{f(a+h)-f(a)}{h}$ when the output of the limit seemed strange. For instance, for the function x

↦ |3 x-5 |, this limit was -3.*sign*(-3*a*+5) and students did not know what the function *sign* was. The teacher had to tell them to try values. They also graphed the function and recognised that the derivative is -3 for *a*<5/3 et 3 for *a*>5/3. Many students thought that for *a*=5/3, the derivative was zero, because they saw the *x* axis tangenting the curve. A hard discussion was necessary to make clear that there is actually no derivative.

Discussion

In the above hypothesis students using CAS can build deep links between their enactive knowledge and computable representations, first by working on several representations and on a large set of functions, then by looking carefully at the expressions.

In the sessions students were able to see how an analytic approach was necessary to get a consistent idea of the tangent line. In this process, the graphical view was interesting, because the idea of a line "fitting a curve" is best expressed graphically, but also because this graphical representation cannot be used to discriminate between two lines "fitting well’. After the elaboration of an analytic expression of the slope of a tangent line, students could use the power of CAS to make many similar calculations. So students could say: "We did not know at first what exactly the tangent was. Expressing analytically the problem of a line of best fit, we get means to compute the slope of the tangent line of many curves, and then, we had a more general idea of the tangent line."

Their work on this expression is worth a reflection. In the first session, the calculator was used just for graphing. All calculations were done by hand. This choice of the teacher not to use the calculator might seem very cautious. On the other hand, a formulation like y1(1+h)-y2(1+h) (y1 and y2 being respectively the name of the function and of the right member of the equation of the line) is not familiar for 11th grade students. They were not able to express the distance directly with this formulation and preferred to use a sketch to think about it. Doing this, they transformed the expression and finished the calculation by hand. An other factor for not using CAS was that they were interested by the reason for the cancellation.

In the other sessions, CAS was used extensively. Students had to adapt their paper/pencil procedure into actions with the TI–92. At first, the actions were three successive simplifications of expressions: the distance, the quotient, the limit. Progressively, students made a single expression. In the third and fourth sessions they used the parametric expression of the derivative built with the teacher, but came back to simplifying the quotient when the output of the limit seemed strange. This process cannot be thought just as "first solve by hand then translate into CAS". Qualitative improvements were done inside CAS. Passing from a point to another, then from a function to another, students had to adapt successively the expressions. This task was not so easy and forced students to look carefully at the expressions. Moreover, the generalisation and encapsulation of the search of "best slope" into the parametric expression of the derivative was done without pencil and paper work.

So paper and pencil helped students to understand the calculation, while the work inside CAS gave additional meaning to this calculation: expressions were materials that students were able to handle. Of course, this approach does not remove the difficulties: when a function had no derivative at a point, the expression that students obtained did not give them miraculous insight into this new situation. In the example of the derivative of x ↦ |3 x-5 |, this expression included the *sign* function that students did not know. *sign* is close to the French *signe* and students easily associated this name with the property of a number of being positive or negative. On the other hand, they did not think at first of this property as a function. So, the language of CAS gave students something new to think about, and helped them to reject the horizontal tangent that they saw on the graph, but a specific reflection and discussion was necessary in order that they understand this new situation.

The upshot is that in a situation like this the expressive power of algebraic expressions exists. This power is a property of CAS: using CAS the expressions generated help students to examine puzzling phenomena occurring in the graphical window, they can be constructed and de-constructed, they can be adapted for new points and functions and generalised into parametric expressions. However, this property would do little for students' understanding without a set of tasks carefully planned and without classroom discussions. Like learning in micro-worlds, learning in CAS settings requires the adequate organisation and the management of didactic situations by the teacher.

Another concluding remark is that the two dimensions of mediation by CAS that I tried to study in this paper are complementary: the assistance of the language and other representations is effective in learning sessions, while time is needed for the development of sets of techniques, richer than just paper/pencil and fostering a conceptual reflection.

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FORUM 1/REACTION

Response to Jean-Baptiste Lagrange: “A didactic approach of the use of CAS to learn mathematics”

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This paper focuses on "the role of techniques in the learning of mathematics", and "the search for conditions in order that the changes introduced with CAS bring satisfactory results". The author points out that school mathematics topics consist of a set of tasks, techniques and conceptual reflections, and he argues that reflection operates on techniques, not on tasks. Thus teachers using CAS to replace the techniques are not succeeding in improving students’ conceptual reflection. Some years ago I commented rather naively on the potential role of CAS in such problems, and suggested that it offered a new vision of mathematics education (Lerman, 1993). Jean-Baptiste’s paper enables me to re-examine my dreams from a more informed position and I am therefore grateful for the opportunity to read and comment on his ideas.

The problem of algebraic manipulative technique is a familiar one in mathematics education and is especially acute for students’ identities as successful school mathematicians where these skills are not strong. An example of this is when students lose sight of the goal of mathematical induction whilst struggling with the algebra necessary to prove the case for $k+1$ having assumed the case for k. Another, but different, instance occurred recently. Some months ago I observed a 13 year old student resisting engaging with a set of examples of factorising algebraic expressions because, she said, "I don’t understand what it is all about, and I won’t do anything that I can’t see the reason for." Just two weeks ago I returned to observe that same class and found that student factorising expressions and expanding brackets without hesitation. When I asked her what had happened she said, "Oh well, I still don’t understand what it’s for but I have realised you are not supposed to." The second example concerns meaning and motivation, and I believe this is related to task as well as to technique. I will return to that issue below.

The author offers an insightful example of the didactic problem that he is addressing with the factorisation of x^n-1 where the CAS masks the direction of a solution for some students. The second example, that of the derivative of the product of two functions $u(x)=e^x$ and $v(x)=x^2+x+1$ demonstrates some other students using CAS to assist in the general solution. He suggests that tasks "have a value as a mathematical activity only if students are able to link these results with techniques of symbolic representation". He argues that the expressive power of the CAS differs from microworlds in that expressions are not already known entities such as turtles but are algebraic objects. He draws on a distinction between enactive actions, which are actions on common knowledge, and theoretical actions, which work on expressions of functions, on graphs etc. The study he describes concerning the tangent to a curve at a point is used to illustrate his argument.

I want to bring into this response two other technologically-focused mathematical activities, because I believe there is something similar to them all. The first is a Graphical Calculator activity I have used as an introduction to differential calculus. Through both zooming in and observing local straightness of curves and through successive calculations of the gradients of chords from a point to neighbouring point as the latter approaches the former, students were able to spot a pattern in the derivatives of x^n. When this was followed by performing the same process analytically, using h as the horizontal distance, it led my students to the same position as the author's students. Of course Tall's Graphical Calculus provides a similar approach. The second example is of the use of a Graphical Calculator to find maxima and minima, through the transformation of parabolas or cubics by translation in the direction of the axes. These activities have the same features as those in Jean-Baptiste's paper; through working on tasks in a manner that parallels the analytical, followed by repeating the same technique in general terms, students can reflect on the task and the technique to see and appreciate the generality. Such activities have been described as 'instructive representations' in contrast to 'constructive representations' (O'Reilly, Pratt, & Winbourne, 1997). This is not to denigrate them; on the contrary I am sure they have an essential place in the teaching and learning of mathematics and, as I have said, I use them in my own teaching. I wish merely to add to Jean-Baptiste's 'classification' of technological tools into those that act on already known entities such as turtles, and those that act on mathematical objects. The use of technology in the kinds of examples above is enormously powerful in enabling students to retain the sense of the task whilst gradually engaging with the techniques, in a manner that does not become mere repetitive skills practice, and does not lead to them losing their way. The proof by induction instance I mentioned earlier can result in such problems. Whilst some of my students may be able to state the inductive step (in the 'Handshake' problem to state $[k(k-1)]/2 + k$ for $(k+1)$ people), the simplification process means that they can become lost in their struggle with the technique. Perhaps one would say, however, that if students only engage with instructive representations then they are missing something; this comment is along the lines of DiSessa's point about reading and writing.

I want to return finally to the issue of meaning and motivation. This is not the focus of Jean-Baptiste's paper of course but is of concern to a more general analysis of teaching and learning. I want to say that there is no such thing as a meaningless activity. The student I observed and described above, found meaning both when she could not engage with the task and later when she could. At first she saw the task as a having a meaning and purpose which she could not recognise; later she saw the task as, perhaps, typical of much school mathematics, unconnected with anything in which she was interested but as 'school maths' which has to be done. Jean-Baptiste's paper demonstrates the multiple representations that the CAS offers, especially when combined with others such as Cabri. It shows also the possibility for assistance with techniques that can help the student initially to see through the techniques to the solution of the task, and later to see the techniques as expressing the solution. Thus such activities in the environment can bring the link that Jean-Baptiste seeks, between enactive knowledge and computable representations, and the one that I seek in addition, between useful and motivating tasks and the kinds of mathematical activity that can lead to solutions.

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