FORUM 4/PRESENTATION
Didactical use of CAS in story problems

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Abstract

Here we present an approach to study mathematical models using a Computer Algebra System (CAS). Pupils (ages 13-14, n = 141) were introduced to the mathematical model of the classical problem "How long did Diophantus live". They studied the model by solving problems, manipulating the parameters, and inventing similar story problems. Consequently the pupils found that the model is more than just an equation and that the implicit restrictions are an integral part of the model. Through the interaction of pupils with the computer the teachers became aware of some of the cognitive processes involved in the modeling activities, which in turn made them rethink their teaching methods.

Introduction

Probably all generations of pupils who studied algebra encountered the famous problem from Greek Anthology, "How long did Diophantus live". We know that Diophantus (born about 250 AD) solved arithmetical problems, not equations; however, he was interested in *exact* rational solutions. Therefore, his title as the father of algebra is a matter of historiographical study (Boyer, 1968, pp. 196-216). Father of algebra or not, his place in algebraic textbooks is secure.

Algebraic word problems are notoriously difficult to solve for most pupils. The challenge for educators is to somehow overcome the difficulties. We describe here a few attempts that are relevant to our approach. Stacey and MacGregor (1995) used different verbal descriptions of the same problem to encourage pupils to form mental representations. They reported that many pupils, as they worked, extended their mental models to encompass additional features of the mathematical structure. Hoz et al. (1997) studied in depth the role of structural and semantic factors in solving speed problems. They recommended that teachers use similar or isomorphic problems to clarify the inference of the relations embedded in the structure of the problems.

Influenced by cognitive science, educators have attempted to use computers to improve word problem solving. One such system, *Animate* (Nathan, 1992) creates animations of the story problem derived from algebraic equations constructed by the pupils. Experiments with *Animate* indicated that the animated simulations of the problem provided experiential feedback for error detection and correction (Nathan, 1998). The graphic features of computer software led to a functional approach to story problems as was implemented in *The Algebra Sketchbook* (Yerushalmy & Shternberg, 1992), and in other innovative curriculum projects such as the *Computer-Intensive Algebra* curriculum (Heid, 1996). In addition to the graphic tools, students in a computer-intensive algebra course have access to a symbolic manipulation program during every class. In this paper we approached story problems in the standard algebraic representation using a Computer Algebra System.

Computer Algebra Systems (CAS) first appeared on personal computers in the mid-eighties, opening the way to new teaching strategies and curricular developments. However, research on the use of CAS in mathematics education is relatively new, and most of the work in this area has concentrated on the student-knowledge-technology triangle. Computer algebra software does not answer questions; rather it reacts to an action and produces something that needs to be interpreted. Therefore, it is essential to teach pupils how to make the most effective use of the output of the software (Hunter et al., 1995). A CAS, by freeing students from syntactic manipulation, is thought to allow them to concentrate on semantic or conceptual aspects of algebraic reasoning. Pozzi (1994) studied this in a qualitative case study with students of ages 16-17. He pointed out that students who do not fully comprehend the CAS output would develop informal and possibly erroneous ideas of what the computer is doing. Pozzi concluded that using a CAS might necessitate a better conceptual understanding of the algebraic manipulations. With a CAS executing the procedures efficiently, we can now reflect on relationships that connect the formal notation and the procedures for performing mathematical tasks (Zehavi, 1996, 1997). Studies on learning with a symbolic manipulator have indicated that the teacher plays a far more complex role while teaching in this new environment (Heid, 1996). Drouhard (1997) presented a social aspect of CAS in the classroom. He generalized the *didactic triangle* (student-knowledge-teacher) by defining the *double didactic pyramid* with two new vertices, the CAS and the group. In this study we attempted to investigate the construction of mathematical knowledge related to a family of story problems within a curriculum development project.

The MathComp project

The MathComp (mathematics on computers) project began in 1996 in the Science Teaching Department at the Weizmann Institute of Science with the aim of integrating CAS into teaching, to improve the learning of mathematics with the following goals:

1. Creating a network of relations between mathematical ideas, concepts and the procedures leading to them. This is possible because CAS frees the student from the tedious work involved in carrying out the procedures;
2. Enriching the mathematical language used by students while they “do” mathematics;
3. Developing independent and critical thinking especially while using CAS;
4. Using technology to stimulate and motivate students to do mathematics.

These goals are adequate for any technology-based educational project. However, what really matters is finding the ways to achieve these goals. We decided to develop *mathematical learning units with computer algebra* that accompany the syllabus for junior high school mathematics, which are completed in a computer lab. Each of the units that were developed includes a variety of tasks aimed at improving skills, understanding concepts, and investigating problems. These tasks create a network of mathematical connections, which in turn, give deeper meaning to the subject at hand and open windows for new mathematical experiences.

The basic assumption in planning the MathComp units is that presently pupils do algebra with paper and pencil in class, and use CAS in the lab to broaden learning opportunities and to promote mathematical understanding. This assumption is influenced by practical considerations, as well as by the current pedagogical knowledge of the use of CAS in education. The topics that are dealt with in the units were selected by considering the potential use of *Derive* (Soft Warehouse) to achieve new didactic goals and develop new learning strategies. One of our main concerns was that the *complexity* of the CAS would *simplify* those areas that cause students the most trouble. Here we present a unit for Grade 8 “Equations and problems”, explaining its rationale and the formative development process of the tasks. We describe an exploratory study in four classes that worked on the unit, and discuss the lessons learned and their significance in preparing an updated version of the unit.

Equations and Problems: A learning unit with *Derive*

The MathComp book for grade 8 consists of 11 units, each of which should take 2-3 periods in a computer lab using *Derive*. “Equations and Problems” is the 4th unit. It deals with building a model (equation) for a ‘story’ problem. Clearly, CAS cannot translate a story into an equation, so how can a CAS support modeling activities? Our idea was to choose story problems that do not fall into the common categories (e.g., velocity problems, mixture problems) for which teachers and their pupils ‘pretend’ to have algorithms for solving the problems. The old puzzle about the *School of Pythagoras* (*Problem 1*) seemed suitable for our needs. The mathematical model for the story is quite simple, and thus can be introduced at an early stage and it should influence the learning and teaching of further models. Moreover, this problem includes fractions, which are both intimidating and difficult for pupils to calculate. Fractions are not popular among pupils and teachers; however, they provide a rich network of mathematical ideas. The idea was that pupils will interact with the software by using the *solve* command and other commands to get feedback and to debug their models.

Problem 1: The school of Pythagoras

Pythagoras, who lived in the sixth century BC, ran a school. He was once asked how many students are in his school. After thinking for a while, he said:

1/2 of the students are now participating in a mathematics class.

1/4 of the students are now in a science class.

1/7 of the students are now silently exercising their minds.

In addition to all the above, three girls are walking in the garden.

How many students were in the school?

After solving a problem, we do not proceed immediately to another problem; rather, we challenge the pupils to invent similar word problems of their own. The main goal of such an activity is to shed light on the equation itself as an object for exploring - not just as a tool for finding an answer to a problem. In designing similar problems, one is forced to realize that there are implicit restrictions in the model, in addition to the equation itself.

We have observed that pupils are motivated to play ‘teacher’ and to express themselves by inventing problems that they can identify with. However, in the first experiments we realized that the pupils write whatever story comes to mind, sometimes without writing the algebraic model. Those who constructed equations and applied the *solve* command, often got ‘unreasonable’ solutions (e.g., a negative solution for the equation **x/2 +x/3 + x/4 + 5 = x**).

We therefore need to prepare pupils to reflect on their stories and models. Thus we added three tutorials for making explicit the implicit restrictions of the model.

Tutorial (a) aims to help pupils realize that the sum of the fractions should be less than one. For example, The school's secretary reported that $1/2$ of the students are doing math, $1/3$ are doing science, $1/6$ think in silence, and 3 girls are walking in the garden. The model that describes this story simplifies to x + 3 = x, which evidently has no solution.

Tutorial (b) aims to help pupils realize that the total number of students is proportional to the number of the girls in the garden. For example, *How do we change the problem so that the number of students in the school will be 280 instead of 28?* In fact, this question was suggested by a pupil who discovered that if 30 girls are walking, then the total number of students is 280.

Tutorial (c) aims to help pupils realize the Diophantine nature of the problem. For example, The total number of students in the school is 120; complete in various ways the following report: $1/2$ of the students do math, $1/4$ do science, ___ think in silence, and ____ girls are walking in the garden.

The unit includes a built-in assessment tool based on the famous story about Diophantus (*Problem 2*). Note that we modified the authentic story, in which a son was born and later died, because we realized that the context of pupils' stories was affected by this tragic fact. The model is similar to the previous one, only a little more complex. Both models simplify to an equation of the form (25/28)x + "number" = x; our intention is that pupils will see the relation between the equation $\frac{25}{28}x + n = x$ and a family of stories. Again, pupils are encouraged to write their own stories.

Problem 2: How long did Diophantus live?

A puzzle, similar to the following, was written on the grave of Diophantus:

> Diophantus spent $1/6$ of his life as a child, $1/12$ as a young man, and then $1/7$ of his life as bachelor.
>
> Five years after he got married he left his hometown.
>
> He returned to his hometown 4 years before his death.
>
> Diophantus stayed away from his hometown ½ of his life.
>
> *How long did Diophantus live?*

The next stage in the formative development was an exploratory study, which is described in the following.

An exploratory study

The structure

Step 1: Four experimental Grade 8 classes (n = 141) worked on the first three *Derive*-based units.

Step 2: The teachers of the four experimental classes were asked to write a short essay on how they would have presented *Problem 1* in their classes (without the computer).

Step 3: Two control classes (n = 62) worked on *Problem 1* for 15 minutes with no explanation. The teachers collected the pupils' work, checked the solutions and reported the findings.

Step 4: Period 1 - The experimental classes worked in the lab on *Problem 1.* The pupils were asked to explain, in writing, how they used *Derive.* They also worked on tutorial (a) and (b). A short discussion was held in the lab after each tutorial. Pupils' worksheets and *Derive* files were collected.

Step 5: The four teachers met to discuss the findings of period 1.

Step 6: Period 2 - The experimental classes worked in the lab on tutorial (c). A short discussion was held, and then the pupils were asked to invent a story problem "How many students...". Several stories were presented in class.

Step 7: Period 3 - The pupils of the experimental classes solved *Problem 2* in the lab. They invented story problems "how long...". Pupils' worksheets and *Derive* files were collected. Several problems were presented in class.

Step 8: The teachers met to discuss the findings from period 3 and to make suggestions for improving the unit.

Findings and discussion

Step 1: The teachers of the experimental classes integrated the MathComp units in the teaching of algebra. The pupils became familiar with the following computer procedures: substituting numbers in expressions, solving (in)equalities, and performing operations on both sides of (in)equalities.

Step 2: The four experimental teachers described the following teaching strategies in their essays. The teacher of class A wrote that she would just give them the problem and after a while (depending on how the class behaves) ask several students to present and explain their answers. The teacher of class B wrote that she would first remind the class how to solve an equation with fractions (multiplying by the common denominator). She thought that this would give them a hint for constructing the needed equation. The teacher of class C wrote that she would start by introducing a simpler problem, for which the equation is something like $(3/5)x + 14 = x$. The teacher of class D wrote that she would first spend some time analyzing the problem with her pupils, helping them to realize that X is the "whole", meaning that some parts of X are subtracted, and there are 3 left.

Step 3: In the control classes only 24 pupils (out of 62) got the correct answer after working on the problem about 15 minutes with paper and pencil.

Other pupils had the following difficulties:

20 pupils wrote a correct equation but could not debug their wrong solutions;

12 students made ‘reasonable’ mistakes in the equation (e.g., x/2+x/4+x/7+3x=x);

5 of them corrected the equation, but got stuck in the solution;

6 pupils made ‘unacceptable’ mistakes (e.g., multiplying the given fractions).

Step 4: All four teachers were impressed that after only about 10 minutes, all the pupils managed to get the correct answer (“28 students”); some of them got help from the teacher or from their classmates.

We show three lines from a *Derive* file created by one pupil in class C:

#7 $x - \frac{x}{\mathbf{2}} + \frac{x}{\mathbf{4}} + \frac{x}{\mathbf{7}} = \mathbf{3}$ *User*

#8 $\frac{\mathbf{25}x}{\mathbf{28}} = \mathbf{3}$ *Simp(#7)*

#9 $x = \frac{\mathbf{84}}{\mathbf{25}}$ *Solve(#8)*

It was clear to the pupil that his answer must be wrong! He reflected on his work and concluded that the source of the mistake was not in line #8 or #9. Thus, he modified line #7 several times until he got it right.

Table 1 [below] summarizes pupils’ work on *Problem 1* based on what they wrote in their worksheets and in *Derive* files that were created during their work. Row 1 shows the two basic types of equations that the pupils wrote in attempting to solve the problem. The first is a straightforward translation of the story, and the second is the result of some processing that pupils did in their heads. (It is not surprising that the pupils who constructed the second model could later explain more easily than the others that we get the solution, x = 280, if we change the number of girls from 3 to 30). The last row presents the percentages of pupils who concluded their answers, without being explicitly asked, along with the calculation of the number of students engaged in various activities of the story. We will see later the importance of this calculation for expliciting the implicit restrictions on the model.

Step 5: We discussed with the four teachers the distribution in each class and the differences between the classes. We also examined what the teachers wrote before the lesson. The teachers felt that pupils’ work on the problem using *Derive* is somehow related to the teacher’s didactical strategies, techniques, and goals in teaching algebra in the class. Thus, the teachers could draw conclusions regarding their teaching. The different percentages of the occurrence of the two types of equations could have been caused by the differing types of guidance provided by the teachers for constructing models in previous algebra lessons. It was also clear that class B had been instructed to ‘get rid of the denominators’. However, when using CAS, pupils can see better the structure of the model when they simplify the expression with fractions. The different percentages of pupils, who calculated the number of students engaged in each activity, definitely indicate the routine maintained by each teacher.

Step 6: Tutorial (c) created great interest in class D. The pupils were not satisfied with 2-3 different solutions (as was the case in the other classes). They exhausted all the possibilities for filling in the missing numbers.

Table 1: Distribution of solutions to Problem 1

	Solution to Problem 1: School of Pythagoras	Class A n = 38	Class B n = 33	Class C n = 36	Class D n = 34
EQ	$\frac{x}{\mathbf{2}}+\frac{x}{\mathbf{4}}+\frac{x}{\mathbf{7}}+\mathbf{3}=x$	84%	100%	75%	60%
EQ	$x-x(\frac{\mathbf{1}}{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{4}}+\frac{\mathbf{1}}{\mathbf{7}})=\mathbf{3}$	16%	0%	25%	40%
PROCEDURES	Solve, (Simplify)	35%	45%	30%	73%
PROCEDURES	**Simplify & Solve**	60%	15%	40%	18%
PROCEDURES	Operations on both sides, Simplify, (Solve)	5%	40%	30%	9%
Calculation		5%	3%	33%	20%

Step 7: The number of complete story problems (i.e. story, equation and reasonable solution) in each class varied from 33% to 61%. To illustrate the most common construction of a story and its model by the pupils, we present the story of Uncle David's birthday (*Problem 3*), and the associated *Derive* file (Table 2).

Problem 3: Uncle David's Birthday

On his birthday, Uncle David challenged his friend with the following puzzle:

I spent ½ of my life in a Kibbutz.
I served 3 years in the army.

I stayed in Los Angeles for ¼ of my life.
I traveled in India 3 more years.

I returned to Tel Aviv and so far I have lived there 1/7 of my life.

How old am I today?

Table 2: Constructing the problem of Uncle David's birthday

1	2
#1: "Uncle David's Birthday"	#9: $\frac{19\cdot x}{20} + 7 = x$
#2: $\frac{1}{2}\cdot x + 3 + \frac{1}{3}\cdot x + 4 + \frac{1}{4}\cdot x = x$	#10: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 4 + \frac{1}{7}\cdot x = x$
#3: "solve"	#11: $x = \frac{196}{3}$
#4: $x = -84$	#12: $\frac{25\cdot x}{28} + 7 = x$
#5: "simplify"	#13: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 3 + \frac{1}{7}\cdot x = x$
#6: $\frac{13\cdot x}{12} + 7 = x$	#14: $x = 56$
#7: $\frac{1}{2}\cdot x + 3 + \frac{1}{4}\cdot x + 4 + \frac{1}{5}\cdot x = x$	
#8: $x = 140$	

The file clearly illustrates the monitoring, evaluating, and planning processes that constituted the pupil's reasoning. We can see that he used the *solve* command for monitoring the model and the *simplify* command for evaluating the expressions. Apparently, he understood that the sum of the fractions should be less than 1 (see line #7), but the role of the "number" in determining the solution was not clear to him (see the trial and error in lines #10 - 12).

Step 8: The story problems that students wrote after *Problem 2* served as a source of insight to the teachers and to the project team. The pupils were invited to present their problems in class. The stories composed by the pupils reflected their genuine concerns and attitudes, which kept them highly motivated. Everyone who had what they called a 'correct' problem wanted to share it with the group. A few argued that *their* problems were really interesting, whereas the *given* problems dealt with 'strange' people like Diophantus. We noted this comment.

We present here two more examples. One girl applied a less common strategy in the construction of *Problem 4*.

Problem 4: Math & Music

1/9 of my life I solved math exercises by hand.

Then I was introduced to *Derive* and used it for 1/3 of my life.

Later, I knew everything, so I did math in my head for another 1/3 of my life.

In the last 26 years I did not do any math, I just played music.

How long have I lived?

She explained that because she is 13 years old, she chose 1/9 so that the solution will be close to 120 (i.e. 117). Then she calculated the numbers to obtain an equation:

$$\frac{x}{\mathbf{9}}+\frac{x}{\mathbf{3}}+\frac{x}{\mathbf{3}}+\mathbf{26}=x\ .$$

Not all the presentations in the classes went smoothly. When one pupil presented a story problem associated with the equation $\frac{x}{\mathbf{2}}+\frac{x}{\mathbf{3}}+\frac{x}{\mathbf{10}}+\mathbf{7}=x$ and solved it, x = 105, the group noticed that the substitution of 105 does not yield integer numbers in some parts of the equation. This started an argument in class, creating difficulties for the pupils and the teacher, who tried to explain the problematic cancellation of fractions.

Consequently, the teachers suggested adding a question to tutorial (c) to deal with the restriction on the solutions of Diophantine equations:

During a school basketball game, four players scored all the points.

Dan scored 40% of the points, Ron scored 25%, Eyal scored 15%, and Oded made five 3-point baskets.

How many points did the team score? How many points did each player score?

The problem about scoring points in a basketball game should be attractive to the pupils. This context enabled us to deal with fractions in the form of percentages; thus the problems do not look exactly the same as before.

Conclusion

In this study we found that the use of technology for solving word problems does not replace reflection on the part of the student, but rather enhances it. When pupils are using CAS to solve a word problem and immediately get a solution, they are free to check if the solution does not make sense. In such a case they know that the source of the trouble is the model (equation) and not the algebraic manipulations. They may reflect on the way they constructed the model and try to resolve the problem. This is a big advantage over the conventional way of teaching, where pupils do not have any immediate control over their work and therefore have to wait until the teacher reacts to their solution before they start to reflect, if at all.

We have emphasized that usually the complete model of a word problem includes, in addition to the equation(s), some implicit restrictions. Because the pupils were asked to play the role of the teacher by inventing their own story problems, they realized the importance of those implicit restrictions. However, we observed that in constructing problems, the proper balance between student interaction with the CAS and student reflection was not always maintained. A significant consequence of our observations and discussions with the teachers is the need to add a *situation of institutionalization* (following Brousseau's theory). This situation will concentrate on the general parametric equation of the model and its symbolic solution, which can be interpreted in terms of the implicit restrictions:

$$\frac{a}{b}x + n = x$$

$$x = \frac{bn}{b-a}$$

The design of the unit leads in a natural way to parametric equations. We are currently working with teachers and pupils to determine how the parametric terminology can be used to summarize the important mathematical ideas, concepts, and procedures of the learning unit.

The approach implemented in the learning unit, *Equations and Problems* stimulated teachers to design their own units for teaching word problems. Our experience with this unit (and others) indicated that using CAS technology increased teachers' awareness of the cognitive aspects of learning. The teachers were challenged to rethink curricular and didactical aspects of mathematics learning. It is our hope that teachers will develop independence and creativity when incorporating CAS in their professional lives.

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FORUM 4/REACTION
A reaction to the presentation "Didactical use of CAS in story problems"

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Analysis of the research paper

The research presented in the paper by Nurit Zehavi and Giora Mann is concerned with the students' processes of solving word problems in a learning environment in which a CAS is available. In this environment, the computer system serves as a tool in the solution process; it does not organize this process as it is the case for environments studied by, for example, Hershovitz and Nesher (1999). The learning process is organized through a careful design of problems and tasks for the students as a group and planning of the activities with the teachers. The aim of the teaching design is to teach mathematical thinking and not to teach mathematical thinking in a computer environment. Mathematical thinking is meant to be based on a network of concepts, procedures and relations between them, expressed in a rich mathematical language. The availability of a CAS is, for the designers, instrumental in reaching this aim, by freeing the students from some mechanical algebraic and numerical manipulations and allowing them to concentrate on mathematical ideas and relations between them. The authors are aware of the necessity of preparing the students to use computers in an intelligent way (critical and independent). They are aware of the fact that a sensible, critical and independent use of computers in learning mathematics may require a far deeper understanding on the part of the students, some of whom might otherwise go through their mathematics education with a few procedural skills. In the design of their teaching experiment, the authors are trying to inscribe themselves into the existing school conditions. In particular they do not try to keep all classroom activities in a computer lab. There are traditional classroom sessions, with paper and pencil, and the computer lab is used for enrichment activities.

The most interesting feature of the project presented by the authors is that the students' task is not only to solve problems given by the teacher but also to invent problems. I propose that it is this feature of the didactic situation and not so much the availability of a CAS that was responsible for the students' progress in their thinking about modeling word problems using equations and conditions on variables.

In this situation the students owned some means of control over their work; the validation of the results of their work did not rest solely with the authority of the teacher. The CAS that the students could use was affording some help in the verification, but I consider the ownership of the means of control as a factor of progress, not the presence of CAS.

In general, I believe that tasks in which the students have no control over the results of their work are destructive for the learning of mathematics. An example of such a 'destructive task' could be the following: 'Given $f(x) = \frac{-3x-2}{-x-4}$ and $g(x) = x^2 + x - 1$, find $g(f(x))$. Simplify your answer'. If a CAS was available for the students, they could just type in the two functions, obtain their composition and ask the computer to simplify. The means of control, in this situation, would be embodied in the computer program. Without a CAS, the only way for a student to verify the result would be to re-do the calculations, and wait for the verdict of the teacher. The message that the students receive through being given problems such as this is that validity in mathematics (a) is decided by the teachers, and (b) is reduced to the notions of correct, partly correct and incorrect answer. The students, trained this way, take no responsibility for the answers to the problems they are given. They only take responsibility for knowing the rules of the didactic contract. In fact, the most frequently asked questions in mathematics courses are related to the presentation of solutions (and not, for example, to the interpretation of the representations of mathematical objects): e.g. 'do *you* (the teacher) want us to reduce everything to a common denominator, or can we leave the answer as the sum of two fractions?. Are we going to get some partial marks or is it just all or no marks?, etc.'.

Problems in which the students have to solve an equation are better from this point of view: the students can verify if the number(s) they obtained satisfy the equation. However, if the equation comes from the teacher, and/or the students do not perceive it as a condition on variables, the availability of the means of control may not translate into an ownership of the means of control for the students. But in the situation use in the experiment, by 'playing the teachers', the students were taking on some of the responsibility for the validity of the knowledge produced in the classroom. In the classroom discussions of the problems invented by the students, the teacher and the students were in the roles of colleagues or partners, and mathematical, not didactic, arguments had to be used in the evaluation of the proposed problems.

Moreover, the process of setting up a problem and evaluating it required that the students take a theoretical perspective on solving equations; they could no longer think about equations as expressions to be transformed to expressions of the form x=..., y=..., etc., but as conditions on variables which can be satisfied by some values of these variables. The students also had to take an applied mathematical perspective and think about the domain of validity of an equation not just in terms of algebraic rules such as 'the denominator cannot equal to zero', but also in terms of the quantities that the variables were assumed to represent in the equation viewed as a model of a relation between these quantities. This is distinct from the role that is incumbent upon the students in the traditional contract: the 'theory' belongs to the teacher, the students have to master the 'technical skills'.

The point that I am trying to make is that it is the ownership of the means of control over the results of their work, the responsibility for the validity of the knowledge produced in the classroom, and a theoretical perspective on the subject of study that could account for the progress in the students' mathematical thinking, rather than the availability of a CAS.

General considerations on the role of situational characteristics in the effectiveness of the use of technology in mathematics teaching and learning

I think that, in the research on the use of technology in mathematics education, the question is no longer, 'Does technology enhance students' mathematical thinking?' but 'How are students and teachers using technology in given didactic situations? What mathematics is being taught and learned in these situations?'. The research paper under discussion exemplifies this trend, although, maybe, the features of the didactic situations in which the students were using technology did not receive sufficient attention (or were not sufficiently explicit) in the analysis of the results.

The research on students' mathematical conceptions developed through interactions with computer representations started with an optimistic expectation that these representations will make the abstract mathematical concepts more 'concrete' and easier to understand. We now know how naïve this expectation was. The apparent 'concreteness' of a computer representation has been found to often mislead the students who interpret it as an icon (resembling an object) rather than as a symbol (whose relation to the object is based on conventions and concepts). Our own research (Sierpinska, Dreyfus & Hillel, 1999) on the teaching and learning of the notions of vector and linear transformations in an extended Cabri-geometry environment has surprised us with a variety of unintended yet justifiable interpretations of the Cabri representations by the students.

The designers of computer representations of mathematical objects are fast to assume certain 'basic' understandings. These understandings, in fact, could be 'basic' mathematically, but not from a cognitive point of view. For example, Hern and Long (1991) introduced their geometric representations of linear transformations in three dimensions by the images of the unit cube, by saying: 'we know the image of any vector x under a linear transformation L if we know the images of the standard basis vectors'. Easier said than understood! In spite of being a direct logical consequence of the definition of linear transformation, it was very hard to understand by all students in our research without exception. The notion requires the students to think analytically (in terms of defining properties) about linear transformations, while they normally start with building their understanding on prototypes, or 'typical examples'. It took the best students several weeks of sustained focus on problems related to the construction of linear extensions of transformations of a basis to finally see the above idea. We stress the word 'construction'.

Overall, the problem with technology as a teaching aid could be the same as with any teaching aid: they are properly used only by those students who can do without them. Technology is assumed to help students develop analytic thinking but they need to think analytically in order to use it properly.

What can be done? Perhaps the paradox can be solved by getting out of the vicious cercle of the restricted student-machine relations and into the 'milieu' of the tasks, problem situations and the social group relations.

It may well be that computer representations enhance understanding when provided not by the teacher or a ready-made computer program, but when constructed or programmed by the students themselves. This is what research appears to confirm (e.g. Dubinsky, 1997, where students were engaged in programming in ISETL; Yerushalmy, 1997, where students were constructing representations of functions of two variables).

In a recent paper, Guin and Trouche (1999) proposed a categorization of the styles in which students in their experiment worked with graphic calculators. They distinguished several 'work methods': 'random', 'mechanical', 'resourceful', and 'theoretical'. The 'random work method' was characterized by 'trial and error procedures with very limited references to understanding tools [such as semantic interpretation, coordination of different representations, inference, investigation], and without verifying strategies of machine results'. It would be interesting to know to what degree this behavior could be explained by students' difficulties in mathematics and/or general unfamiliarity with the machine, and what was the impact of the characteristics of the problem situations in which the students were expected to use the machines. It is noteworthy that the problems solved by the students in the research appear rather decontextualized, they were given to the students, rather than invented by them, and chosen so as to be particularly inappropriate for study using a graphing calculator. Here is one of the problems given to the students, as cited by the authors:

> $P(x) = 0.03x^4 - 300.5003x^3 + 5004.002x^2 - 10009.99x - 100100$.
> a. Determine the limit in $+\infty$.
> b. Determine a window which confirms your result. (Guin & Trouche, 1999).

The graphical mode is rather inadequate to study functions such as P(x) above: a fourth degree polynomial with wide ranging coefficients. The formulation of the problem is also misleading, because no graphical representation of a function can 'confirm' its supposed behavior in infinity. In general, the aid of computers in studying the problems of limits of functions requires extreme caution, especially with beginners. It is possible that, in a different situation, the 'random work method' would not appear, or would appear in a smaller number of students.

Some concluding remarks

If it is true that the characteristics of the didactic situation in which technology is used by the students have an important impact on what is being learned, then the fact that we still know very little about these characteristics should make us wary of the hasty curricular changes that are taking place around the world, making it a law to use technology in mathematics education. In fact, researchers have not been recommending curriculum changes (see, e.g. Mann, Rothery and Sato, 1997). Teacher training is often seen as the crucial point. There is not enough research based, positive advice to give teachers concerning the organization of their teaching, and the management of the students' learning in computer environments. The problem is that we know enough to be worried about a mass use of technology in mathematics teaching, but not enough to help avoiding a massive failure of this educational enterprise. Guin and Trouche (1999) report that, in spite of the fact that all students in French secondary scientific classrooms have graphic calculators, only 15% of the teachers include them in their teaching. The resistance of teachers to the implementation of a technology based curriculum is probably more widespread than we can possibly imagine. In Québec, the new textbooks, which, for the teachers, often replace the official curriculum, rely heavily on the use of the TI-83 graphing calculator and Cabri-geometry II. There is a sense of panic among teachers, especially with respect to the idea of using a dynamic geometry software. In some schools one, usually young, new and part-time teacher is planned to be delegated 'to run the Cabri lab'. And what if there is no one willing or able to do this? Well, in that case, the interactive work with the software could be replaced by looking at the pictures of the Cabri screen reproduced in the textbooks. And if this is the case on a larger scale, then it will not be long before we hear about yet another reform of mathematics teaching.

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