CAME Workshop 1999: Some notes on expressiveness

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The research fora at the workshop presented many ideas about the role of CAS in mathematics education. I would like to frame some comments on these in terms of the notion of expressiveness, and in particular what I will call expressiveness "in the local" and "in the global". (These notes are intentionally sketchy—see the above website for later developments.)

The idea of expressiveness

The idea of expressiveness that I'm referring to here has developed amongst the research community associated with the programming language Logo and its descendants (for history and discussion, see diSessa, Hoyles & Noss 1995; Noss & Hoyles 1996). Early on in the history of Logo, the "microworld" was proposed as the prototypical learning environment in which the Logo language would find a role. In Papert's book *Mindstorms* (1980, p. 120), microworlds are described as

> incubators for knowledge . . . First, relate what is new and to be learned to something you already know. Second, take what is new and make it your own: Make something new with it, play with it, build with it.

Much has been said, by friends and enemies of the microworld idea, about these notions of "playing" and "building". For a while in the 1980s, microworld became a buzz word, applied to any piece of software that was (according to some definition) "interactive" and intended for learning. At the level of simple descriptions (which is where, unfortunately, much discussion—especially of a political kind—takes place), the term microworld was both seductive and fatally vague. But fortunately, ideas about Logo (et al) have evolved along a number of important directions; and these, I would say, have not yet been sufficiently recognised in the general mathematics education community.

Firstly, *constructionism* has developed as a theoretical framework for thinking about learning in computational environments (Harel & Papert 1991):

> Constructio*n*ism—the N word as opposed to the V word—shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of the learning. It then adds that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sandcastle on the beach or a theory of the universe. (Papert, in ibid, p.1)

According to Noss & Hoyles (1996, p. 61), the essence of Papert's "constructionist paradigm" is the "dialectic between the design of learning environments and the research effort to describe student's learning within them".

Secondly, the idea of an *expressive medium* has developed as a way of characterising constructionist learning environments. Expressiveness means that it is possible to *express* ideas (mental objects) in concrete form (visible, public objects); moreover, in an expressive medium, actions are carried out by means of programming in a syntactically precise language. Richard Noss and Celia Hoyles have used the term “auto-expressive”, to emphasise the way that a computational medium can be a place where mathematical actions are both performed *and* reflected upon:

> … we have expressed our action in a language which in turn, allows us to reflect on our action. We did not have to leave the paper (metaphorically) on the table and transport our thinking to another, algebraic world of inert symbols: in our dynamic algebraic world the symbols came alive. We have breathed life into the symbols: they come alive because they can be *executed*. (Noss 1997)

So much for the uses of expressiveness by learners. But there is a flip side to this, which is particularly important when considering (in the words of this workshop’s programme) the place of CAS research and CAS-related activities within mathematics education research as a whole. For an observer (teacher, software designer, educational researcher), the concrete, sytactically-precise expression of thought in actions (writing and rewriting of computer code, gestures and spoken remarks with respect to code and graphical and numerical displays) offers an unprecedented “window” onto learners’ mathematical thinking (cf. Noss & Hoyles 1996). I think that this is a point that can have a lot of appeal to the general maths education community, and which we (the CAME community) do not promote enough.

Expressiveness in the local

CAS is often spoken about as providing new routes into existing knowledge. For example, Paul Drijvers (Pre-proceedings, p. 36) mentions the work of Heid on resequencing concepts and skills: “Is it possible to ‘resequence’ a course using a CAS, so that concept development precedes the solving techniques and algorithms?”.

I have no doubt that virtually everyone present at the workshop believes that resequencing is not only possible, but demonstrably proven (the paper Kent, James & Ramsden 1998, describes some of my own experiences in the area of calculus). Whilst reading the Pre-proceedings, I was struck by the following reflection. A “computer algebra system” seems purpose-built for doing algebra, but there are good reasons to say that algebra, as conventionally understood in school mathematics, may be the hardest thing to do with computer algebra! Jean-Baptiste Lagrange (Pre-proceedings, p. 6) offered an example of this:

> These teachers wished that tenth grade students find general factorisations of [the polynomial $x^n - 1$] and regretted that, in the paper/pencil context, students were restricted in their investigation by the difficulties of the calculations and notations. They tried to use DERIVE to liberate students “from the technical aspects of computing by hand” and then “to keep sight of the main goal”.

However, the “main goal” here turns out to involve a piece of advanced undergraduate mathematics:

> [The students] could discover techniques to get factorisations in a number of situations (for instance, when *n* is prime, or a product of two primes). However, a satisfactory ‘theory’ issuing from these techniques (the theory of cyclotomic polynomials) is beyond the reach of tenth grade students.

Now, there is not necessarily a problem here, it may be a good thing for a learner to get a glimpse of the far-off mathematical horizon, but it demonstrates that “resequencing” is not always quite the clear-cut shuffling of a “pack of ideas” that its literal interpretation suggests.

If CAS is to be an expressive medium for teaching and learning algebra, then it's clear to me that we need to fundamentally revise the conventional idea of "school algebra". As Minsky (1994) puts it:

> The problem with children learning mathematics in this [American] culture is that we have a very strange attitude towards it: we don't have any mathematical words. If you look at any other subject in school … by the time the child is through 12 years of school he probably knows a couple of thousand words in each subject … In mathematics the children don't even know the adjective for which situation you use addition. They learn addition, subtraction ... but they don't learn any words about them that would let them think about the subject.

That is, expressiveness requires both a medium for expression and some worthwhile ideas to express oneself about. And this inevitably brings questions of mathematical *epistemology* into the discussion—see diSessa (1995), Turkle & Papert (1991), Noss & Hoyles (1996).

Expressiveness in the global

> *The computer is an expressive medium that different people can make their own in their own way*. (Turkle & Papert 1991)

Over time, I've become more and more convinced that, if CAS is to have a significant impact on mathematics curriculum of schools and universities, we need to be thinking more globally, to contemplate the potential of CAS in the same way that Papert and his fellows were prepared, twenty years ago, to contemplate the potential of Logo (cf. Papert 1980, 1996). That is, how should we think about CAS as an expressive medium for new structurings of knowledge, new kinds of knowledge, and new connections between knowledge domains?[*]

I'll briefly outline one example of what I mean by thinking (more) globally about CAS (for details see Kent & Noss 2000, Kent 1998). In my project at Imperial College, we have been running for a number of years short introductory courses on Mathematica for first-year students in various engineering disciplines (mechanical, civil and chemical). Now, Mathematica is mathematical software, so it's natural to talk about how to do mathematics with it, especially in a short course which needs to put across "the basics". There is so much that seems to need to be discussed (symbolic calculations, numerics, graphics, programming, using the document interface), that it is easy to spend the whole time exhibiting the functionality of the software. However, we have tried very hard in these courses not to do *only* this, by trying to present a range of activities to the students where different aspects of Mathematica are introduced and then applied in *specific* engineering contexts.

This contextualised approach has some important features to do with forming connections between mathematics and engineering. Most obviously, the students are seeing mathematics connected to engineering situations (which is good motivation for the relevance of a piece of mathematical software to "doing engineering"). And in terms of curriculum development, designing these engineering-based activities has required assistance from our "client"

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[*] Papert's (1980) vision for Logo contained an extremely radical streak, in its strong attacks on the basic notions of conventional schooling. Interestingly, though, this apparent attack from *outside* the school system had significant effects at the grass roots *within* the system:

"Despite Papert's uninhibited denunciation of schooling, his ideas have created a considerable following amongst teachers. Their appeal is not as surprising as it may seem... In particular, accounts of the spontaneity and enthusiasm of students in the Logo environment, the rejection of external structures of curriculum and assessment, and the portrayal of success in the most unpromising circumstances, correspond to three of the aspirations [identified] as paramount for teachers: informality in their interactions with students; autonomy in their relations with superiors; individual response and progress on the part of their students. In effect, Logo appears to offer a 'technical fix'; a technological shortcut to these ideals." (Ruthven 1993) engineering academics, thus giving them a significant stake in the course—which would not exist if we just came in as general “Mathematica experts” to teach the course.

For civil engineering, the contexts we chose were to do with structural analysis. The idea of the engineers was that, by letting Mathematica take the mathematical strain, we could help students begin to get a “structural feel” for how structures behave (or, as we might say here, to use the CAS as an expressive medium for thinking about structures). For example, the first Mathematica session for the students was a quick overview of the numeric, symbolic and graphical capabilities of the software, concluding with an exercise in which the students were invited to apply their fresh knowledge to a typical loaded beam problem, such as they meet in their first-year engineering course on structures. The structural feel idea prompted an emphasis not on the mathematics of the problem—which is given to them in full—but on estimating important structural quantities as a load is varied, using whatever combination of graphical, numerical and symbolic methods they chose.

We have developed this contextualised approach across a number of subject areas, and using both Mathematica and Maple. You can view and download the materials from the METRIC web site (http://metric.ma.ic.ac.uk).

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